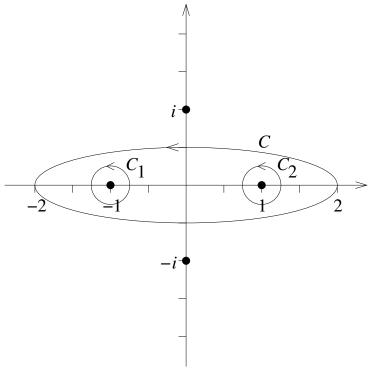
MAT 3321, COMPLEX ANALYSIS AND INTEGRAL TRANSFORMS, WINTER 2005

Answers to Homework 6
13.3 #4,8,18; 13.4 #4,16

Problem 13.3 #4 We will integrate $f(z) = z^2/(z^4-1)$ counterclockwise around the path C given by $x^2+16y^2 = 4$. Note that this is not a circle, but an ellipse with x-intercepts ± 2 and y-intercepts $\pm 1/2$, as shown on the right. Note that $z^4-1 = (z^2-1)(z^2+1) = (z+1)(z-1)(z+i)(z-i)$, thus the function $f(z)$ has 4 singluarities at $1, -1, i$, and $-i$. Of these, only 1 and -1 lie inside C. By independence of path, the desired integral is equal to the sum of the integrals around C_1 and C_2 as shown in the figure. By Cauchy's integral formula, we have



$$\int_{C_1} \frac{z^2}{z^4-1}dz = \int_{C_1} \frac{z^2/(z-1)(z^2+1)}{(z+1)}dz = 2\pi i\, g(-1) = -\frac{\pi}{2}i,$$

where $g(z) = z^2/(z-1)(z^2+1)$. Similarly,

$$\int_{C_2} \frac{z^2}{z^4-1}dz = \int_{C_2} \frac{z^2/(z+1)(z^2+1)}{(z-1)}dz = 2\pi i\, h(1) = \frac{\pi}{2}i,$$

where $h(z) = z^2/(z+1)(z^2+1)$. Finally,

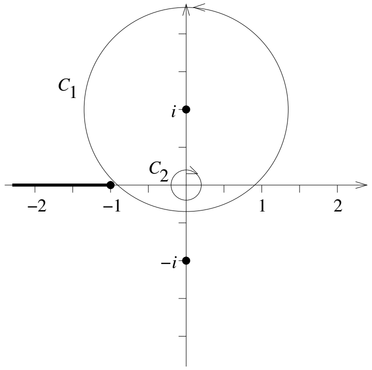
$$\int_C \frac{z^2}{z^4-1}dz = \int_{C_1} \frac{z^2}{z^4-1}dz + \int_{C_1} \frac{z^2}{z^4-1}dz = 0$$

Problem 13.3 #8 We are integrating $f(z) = \dfrac{z^3 \sin z}{3z-1}$ counterclockwise around the unit circle. The function $f(z)$ has a unique singularity at $z = 1/3$. We use Cauchy's integral formula:

$$\int_C \frac{z^3 \sin z}{3z-1}dz = \int_C \frac{\frac{1}{3}z^3 \sin z}{z-\frac{1}{3}}dz = 2\pi i\, g(\frac{1}{3}) = 2\pi i \frac{1}{81} \sin\frac{1}{3} \approx 0.02538i,$$

where $g(z) = \frac{1}{3}z^3 \sin z$.

Problem 13.3 #18 We are integrating $f(z) = \dfrac{\mathrm{Ln}(z+1)}{z^2+1}$ along the path C which consists of $|z-i| = 1.4$ (counterclockwise) and $|z| = 0.2$ (clockwise), as shown on the right. The function $f(z)$ has three singularities at $z = \pm i$ and $z = -1$; the latter singluarity is because $\mathrm{Ln}(0)$ is undefined. Further, the function has a discontinuity along the negative x-axis starting from $x = -1$; this is due to the discontinuity of $\mathrm{Ln}(z)$. Fortunately, our paths of integration do not cross this discontinuity, so we can safely ignore it.



By Cauchy's integral formula, we have

$$\int_C e\frac{\mathrm{Ln}(z+1)}{z^2+1}dz = \int_C e\frac{\mathrm{Ln}(z+1)/(z+i)}{z-i}dz = 2\pi i\, g(i),$$

where $g(z) = \mathrm{Ln}(z+1)/(z+i)$. Therefore, $g(i) = \mathrm{Ln}(1+i)/2i$. Since $1+i = \sqrt{2}e^{\pi/4}$ in polar coordinates, we have

$$\mathrm{Ln}(1+i) = \frac{1}{2}\ln 2 + \frac{\pi}{4}i,$$

therefore

$$\int_C \frac{\mathrm{Ln}(z+1)}{z^2+1}dz = 2\pi i \frac{1}{2i}(\frac{1}{2}\ln 2 + \frac{\pi}{4}i) = \frac{\pi}{2}\ln 2 + \frac{\pi^2}{4}i \approx 1.0888 + 2.4674i$$

Problem 13.4 #4 Using Cauchy's integral formula for derivatives of an analytic function, we have

$$\int_C \frac{z^6}{(2z-1)^6}dz = \int_C \frac{\frac{1}{2^6}z^6}{(z-\frac{1}{2})^6}dz = \frac{2\pi i}{5!}f^{(\mathrm{v})}(\frac{1}{2}),$$

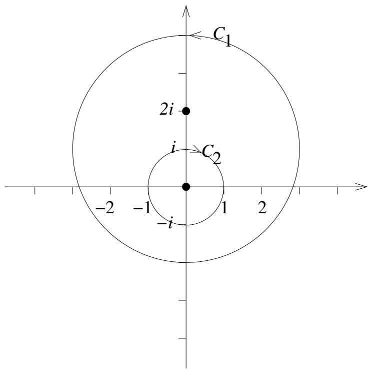
where $f(z) = \dfrac{1}{2^6}z^6$. We calculate this function's 5th derivative:

$$\begin{aligned} f(z) &= \tfrac{1}{64}z^6 & f'''(z) &= \tfrac{120}{64}z^3 \\ f'(z) &= \tfrac{6}{64}z^5 & f^{(\mathrm{iv})}(z) &= \tfrac{360}{64}z^2 \\ f''(z) &= \tfrac{30}{64}z^4 & f^{(\mathrm{v})}(z) &= \tfrac{720}{64}z \end{aligned}$$

Therefore,

$$\int_C \frac{z^6}{(2z-1)^6}dz = \frac{2\pi i}{5!}\cdot\frac{720}{64}\cdot\frac{1}{2} = \frac{2\cdot 720\pi i}{2\cdot 120\cdot 64} = \frac{3\pi i}{32} \approx 0.2945i$$

Problem 13.4 #16 We are integrating $f(z) = \dfrac{e^{z^2}}{z(z-2i)^2}$ along the path C consisting of $|z-i| = 3$ (counterclockwise) and $|z| = 1$ (clockwise), as shown in the figure. Note that the function $f(z)$ has two singularities, at $z = 0$ and at $z = 2i$. Of these, $z = 2i$ has winding number 1 (it lies only inside C_1), and $z = 0$ has winding number 0 (it lies inside both paths, but since the paths are traversed in opposite directions, their effects cancel each other out). The desired integral is therefore equal to integrating around a small circle centered at $z = 2i$, counterclockwise.



$$\int_C \frac{e^{z^2}}{z(z-2i)^2}dz = \int_C \frac{e^{z^2}/z}{(z-2i)^2}dz = 2\pi i f'(2i),$$

where $f(z) = \dfrac{e^{z^2}}{z}$, thus

$$f'(z) = \frac{2z\,e^{z^2}z - e^{z^2}}{z^2} = (2 - \frac{1}{z^2})e^{z^2},$$

thus $f'(2i) = (2 - \frac{1}{-4})e^{-4} = \dfrac{9}{4}e^{-4}$. therefore

$$\int_C \frac{e^{z^2}}{z(z-2i)^2}dz = 2\pi i\frac{9}{4}e^{-4} \approx 0.2589i$$