MAT 3321, COMPLEX ANALYSIS AND INTEGRAL TRANSFORMS, WINTER 2005

Answers to the First Midterm, Version 1

Problem 1. Find the exact solutions of the equation $z^2 + (6i-4)z - 6 - 13i = 0$. The answers must be given in the form $a + ib$, where $a, b \in \mathbb{R}$.

Answer: We use the quadratic formula for $az^2 + bz + c = 0$, which yields the answers as $z_{1/2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. Here, $a = 1$, $b = 6i - 4$, and $c = -6 - 13i$. We find $b^2 = -36 - 48i + 16 = -20 - 48i$, and hence:

$$\begin{aligned} z_{1/2} &= \frac{-6i + 4 \pm \sqrt{-20 - 48i - 4(-6 - 13i)}}{2} \\ &= \frac{4 - 6i \pm \sqrt{-20 - 48i + 24 + 52i}}{2} \\ &= \frac{4 - 6i \pm \sqrt{4 + 4i}}{2} \\ &= 2 - 3i \pm \sqrt{1+i} \end{aligned}$$

We calculate $\sqrt{1+i}$. We have in polar coordinates $1 + i = \sqrt{2}e^{i\pi/4}$, hence $\sqrt{1+i} = \pm\sqrt[4]{2}e^{i\pi/8} = \pm\sqrt[4]{2}(\cos\pi/8 + i\sin\pi/8)$. Therefore

$$z = 2 - 3i \pm \sqrt[4]{2}(\cos\pi/8 + i\sin\pi/8).$$

The exact two solutions are:

$$\begin{aligned} z_1 &= (2 + \sqrt[4]{2}\cos\pi/8) + i(-3 + \sqrt[4]{2}\sin\pi/8) \\ z_2 &= (2 - \sqrt[4]{2}\cos\pi/8) + i(-3 - \sqrt[4]{2}\sin\pi/8) \end{aligned}$$

We can approximate these solutions using calculators:

$$\begin{aligned} z_1 &\approx 3.0986841 - 2.5449101i \\ z_2 &\approx 0.9013159 - 3.4550899i \end{aligned}$$

Problem 2. Determine $a \in \mathbb{R}$ such that the function

$$u(x, y) = e^{2x}\cos ay$$

is harmonic, and find a conjugate harmonic.

Answer: We calculate the partial derivatives:

$$\begin{aligned} u_x &= 2e^{2x}\cos ay \\ u_{xx} &= 4e^{2x}\cos ay \\ u_y &= -ae^{2x}\sin ay \\ u_{yy} &= -a^2e^{2x}\cos ay \end{aligned}$$

So we have $u_{xx} + u_{yy} = (4 - a^2)e^{2x}\cos ay$, which is identically 0 only if $4 = a^2$, or $a = \pm 2$. Since $\cos 2y = \cos(-2y)$, in both cases, the function u is equal to

$$u = e^{2x}\cos 2y.$$

For the following, assume $a = 2$. If v is a conjugate harmonic, then $v_x = -u_y = 2e^{2x}\sin 2y$, hence $v = e^{2x}\sin 2y + h(y)$, where h depends only on y. It follows that $v_y = 2e^{2x}\cos 2y + h'(y) = u_x = 2e^{2x}\cos 2y$, hence $h'(y) = 0$ and $h(y) = C$ is a constant. Therefore,

$$v(x, y) = e^{2x}\sin 2y$$

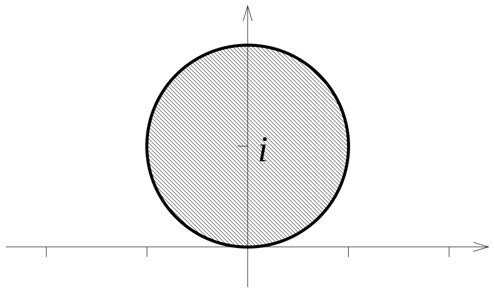
is a conjugate harmonic to $u(x, y) = e^{2x}\cos 2y$.

Problem 3. (a) Sketch the set in the complex plane given by $|z|^2 \leqslant 2\,\mathrm{Im}\,z$.

Answer: With $z = x + iy$, we have $|z|^2 = x^2 + y^2$, hence

$$|z|^2 \leqslant 2\,\mathrm{Im}\,z \iff x^2 + y^2 \leqslant 2y \iff x^2 + (y-1)^2 \leqslant 1.$$

Hence the region D is the closed disc with center $i = (0, 1)$ and radius 1.



(b) Find the image of the region $|z|^2 \leqslant 2\,\mathrm{Im}\,z$ (excluding $z = 0$) under the mapping $w = 1/z$.

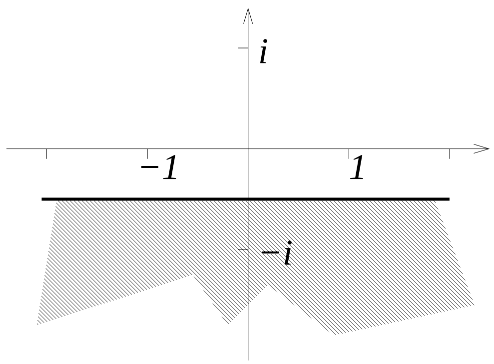
Answer: We calculate $w = u + iv$:

$$w = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2},$$

hence $u = \frac{x}{x^2+y^2}$ and $v = \frac{-y}{x^2+y^2}$. Assuming $z \neq 0$, we have

$$|z|^2 \leqslant 2\operatorname{Im} z \overset{(a)}{\iff} x^2+y^2 \leqslant 2y \iff \frac{1}{2} \leqslant \frac{y}{x^2+y^2} \iff \frac{1}{2} \leqslant -v.$$

The image is therefore the set of points with $v \leqslant -\frac{1}{2}$.



Problem 4. Recall that the complex cosine function is defined as

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}).$$

(a) Calculate $u = \operatorname{Re}\cos z$ and $v = \operatorname{Im}\cos z$. Give your answer in terms of x and y, where $z = x + iy$. Show full details.

Answer: Starting with $z = x + iy$ and the definition of cosine, we get

$$\begin{aligned}
\cos z &= \tfrac{1}{2}(e^{iz} + e^{-iz}) \\
&= \tfrac{1}{2}(e^{ix-y} + e^{-ix+y}) \\
&= \tfrac{1}{2}(e^{ix}e^{-y} + e^{-ix}e^{y}) \\
&= \tfrac{1}{2}(e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)) \\
&= \tfrac{1}{2}\cos x(e^{y} + e^{-y}) - \tfrac{i}{2}\sin x(e^{y} - e^{-y}) \\
&= \cos x\cosh y - i\sin x\sinh y
\end{aligned}$$

Therefore $u(x,y) = \cos x\cosh y$ and $v(x,y) = -\sin x\sinh y$.

(b) Verify that u and v from part (a) satisfy the Cauchy-Riemann equations.

Answer: We calculate the partial derivatives:

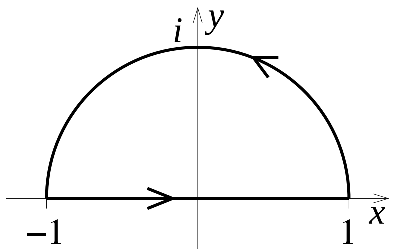
$$\begin{aligned}
u_x &= -\sin x\cosh y \\
u_y &= \cos x\sinh y \\
v_x &= -\cos x\sinh y \\
v_y &= -\sin x\cosh y.
\end{aligned}$$

Therefore evidently $u_x = v_y$ and $u_y = -v_x$.

Problem 5. Evaluate the path integral

$$\int_C \bar{z}\,dz$$

for the path C shown in the figure:



Answer: We parameterize the path as follows:

$$\begin{array}{lll}
C_1: & z(t) = -1 + t, & \text{where } t = 0\ldots 2, \\
C_2: & z(t) = e^{it}, & \text{where } t = 0\ldots \pi,
\end{array}$$

The function to be integrated is $f(z) = \bar{z} = x - iy$, where $z = x + iy$. We calculate:

$$\int_{C_1} \bar{z}\,dz = \int_0^2 \overline{z(t)}\dot{z}(t)dt = \int_0^2 (-1+t)1\,dt = [-t + t^2/2]_0^2 = 0$$

$$\begin{aligned}
\int_{C_2} \bar{z}\,dz = \int_0^\pi \overline{z(t)}\dot{z}(t)dt &= \int_0^\pi \overline{e^{it}}\,ie^{it}\,dt = \int_0^\pi e^{-it}\,ie^{it}\,dt \\
&= \int_0^\pi i\,dt = \pi i
\end{aligned}$$

So therefore $\int_C \bar{z}\,dz = \int_{C_1} \bar{z}\,dz + \int_{C_2} \bar{z}\,dz = \pi i$.