MAT 3343, APPLIED ALGEBRA, FALL 2003

Problem Set 4, due November 4, 2003

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Problem 1. Recall the definition of Euler's φ-function: $\varphi(n)$ is the number of invertible elements in $\mathbb{Z}_n$. Prove that if p is prime, then $\varphi(p^k) = (p-1)p^{k-1}$.

Problem 2. Recall that the Gaussian Integers are the set $\mathbb{Z}[i]$ of complex numbers of the form $a + bi$, where $a, b \in \mathbb{Z}$ and $i = \sqrt{-1}$. In any ring R, we can say that x *divides* y if there exists some $k \in R$ such that $kx = y$. Find all the divisors of $3 + 4i$ in $\mathbb{Z}[i]$.

Problem 3. Let R be a ring with 2 or more elements, which satisfies the *cancellation property*, i.e., whenever $ab = ac$ and $a \neq 0$, then $b = c$. Prove that R is an integral domain (has no zero divisors).

Problem 4. Let $f : \mathbb{C} \to \mathbb{C}$ be the function on complex numbers defined by $f(a + bi) = a - bi$. Prove that f is a ring homomorphism.

Problem 5. (a) Find the inverse of the following matrix with respect to the base field $\mathbb{Z}_5$.

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 4 & 0 \\ 0 & 2 & 1 & 3 \\ 1 & 4 & 2 & 1 \end{pmatrix}$$

(b) Solve the following system of equations in $\mathbb{Z}_3$. What is the most general solution?

$$\begin{aligned} 2x + 1y + 0z + 1w &= 2 \\ 1x + 0y + 2z + 0w &= 1 \\ 0x + 1y + 2z + 2w &= 1 \\ 1x + 2y + 0z + 1w &= 0 \end{aligned}$$

Problem 6. Consider the $(7, 4)$-Hamming code with the following generator matrix G and parity check matrix H:

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Encode the message 0101 1100 1001 0000 1110 using this code.

(b) Decode the message 1100110 1100011 1111000 0111110 1010101, correcting all single-bit errors. What is the syndrome of each codeword?

Problem 7. The $(15, 11)$-Hamming code has parity check matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T.$$

Find the generator matrix for this code. Encode the message 01010101010 11101110111. Find the syndrome of 1000100010001000 and decode.