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Handout 4: The Miller-Rabin Primality Test

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1 Fermat Pseudoprimes

A primality test is an algorithm which, given an integer n, decides whether n is prime or not. The most naive algorithm, trial division, is hopelessly inefficient when n is very large. Fortunately, there exist much more efficient algorithms for determining whether n is prime. The most common such algorithms are *probabilistic*; they give the correct answer with very high probability. All efficient primality testing algorithms are based, in one way or another, on Fermat's Little Theorem.

Theorem 1.1 (Fermat). *If p is prime, then for all $b \in \{1, \ldots, p-1\}$,*

$$b^{p-1} \equiv 1 (\operatorname{mod} p).$$

Definition (Fermat pseudoprime). Let $n \geqslant 2$ and $b \in \{1, \ldots, n-1\}$. We say that the number n *passes the Fermat pseudoprime test at base* b if $b^{n-1} \equiv 1 (\operatorname{mod} n)$. A number n is called a *Fermat pseudoprime* if it passes the Fermat pseudoprime test for all $b \in \mathbb{Z}_n^*$.

By Fermat's Little Theorem, every prime number is a Fermat pseudoprime. Unfortunately, the converse does not hold. There are Fermat pseudoprimes which are not prime. Such numbers are called *Carmichael numbers*. The first few Carmichael numbers are

$$561, 1105, 1729, \ldots$$

Nevertheless, the notion of a Fermat pseudoprime is a useful notion, not least because there is a very efficient probabilistic algorithm for checking whether a given number n is a Fermat pseudoprime.

Proposition 1.2. *If n is not a Fermat pseudoprime, then n fails the Fermat pseudoprime test at base b for* at least half *of the elements $b \in \{1, \ldots, n-1\}$.*

Proof. Suppose n is not a Fermat pseudoprime, and let

$$G = \{b \in \mathbb{Z}_n \mid b^{n-1} \equiv 1 (\operatorname{mod} n)\} \subseteq \mathbb{Z}_n^*.$$

Then G is a subgroup of $\mathbb{Z}_n^*$, thus $|G| \leqslant |\mathbb{Z}_n^*|$. Since n is not a Fermat pseudoprime, there exists some $b \in \mathbb{Z}_n^*$ with $b \notin G$, thus $|G| < |\mathbb{Z}_n^*|$. It follows that $|G| \leqslant \frac{1}{2}|\mathbb{Z}_n^*| \leqslant \frac{n-1}{2}$. Finally, whenever $b \in \{1, \ldots, n-1\}$ and $b \notin G$, then b fails the test; there are at least $\frac{n-1}{2}$ such elements.

Algorithm 1.3 (Fermat pseudoprime test).

Input: Integers $n \geqslant 2$ and $t \geqslant 1$.

Output: If n is prime, output "yes". If n is not a Fermat pseudoprime, output "no" with probability at least $1 - 1/2^t$, "yes" with probability at most $1/2^t$.

Algorithm: Pick t independent, uniformly distributed random numbers $b_1, \ldots, b_t \in \{1, \ldots, n-1\}$. If $b_i^{n-1} \equiv 1 (\operatorname{mod} n)$ for all i, output "yes", else output "no".

Proof. We prove that the output of the algorithm is as specified. If n is prime, then the algorithm outputs "yes" by Fermat's Little Theorem. If n is not a Fermat pseudoprime, then by Proposition 1.2, n passes the test at base b_i with probability at most $\frac{1}{2}$. Hence the probability that n passes all t tests is at most $1/2^t$. □

Algorithm 1.3 can distinguish prime numbers from non-Fermat-pseudoprimes. We did not specify its behavior if the input is a Carmichael number. As a matter of fact, if the input is a Carmichael number, the algorithm will usually output "yes", but will output "no" with a small probability (namely, when n has a common prime factor with one of the b_i).

2 Carmichael numbers

Before describing an improved version of the primality testing algorithm, we prove some useful properties of Carmichael numbers, i.e., non-prime Fermat pseudoprimes.

Lemma 2.1. *Let p^e be a prime power with $e \geqslant 2$. Then the group $\mathbb{Z}_{p^e}^*$ has an element of order p.*

Proof. Consider $G = \{1 + p^{e-1}x \mid x \in \mathbb{Z}_{p_e}\}$. Clearly G is a subgroup of $\mathbb{Z}_{p^e}^*$ with p elements. Since p is prime, each element $g \in G$ has order 1 or p. The only element of G of order 1 is 1, hence e.g. $g = 1 + p^{e-1}$ has order p. □

Proposition 2.2. *Let n be a Carmichael number. Then n is odd, and we can factor $n = m_1 m_2$, where $m_1, m_2 \geqslant 3$ and* $\gcd(m_1, m_2) = 1$.

Proof. To show that n is odd, assume on the contrary that it is even. Then $n \geqslant 4$, since 2 is not a Carmichael number. Moveover, $n-1$ is odd, so we have $(-1)^{n-1} \equiv -1(\bmod n)$. It follows that n fails the Fermat pseudoprime test at base $b = -1$.

To show that n has the desired factorization, it suffices to show that two distinct primes occur in the prime factorization of n. Since n is not itself prime, this is equivalent to proving that n is not of the form p^e, for some prime p and $e \geqslant 2$. Suppose, for contradiction, that $n = p^e$. Then, by Lemma 2.1, there is an element $x \in \mathbb{Z}_n^*$ of order p. Since n is a Fermat pseudoprime, we also have $x^{n-1} \equiv 1(\bmod n)$, hence $p|n-1$. But this is impossible since $p|n$. □

3 Strong Pseudoprimes

Definition (Strong pseudoprime). Let n be odd and write $n - 1 = 2^s l$, where l is odd. Given b, compute the following elements of $\mathbb{Z}_n$:

$$b^l, \quad b^{2l}, \quad b^{4l}, \quad \ldots, \quad b^{2^{s-1}l}, \quad b^{2^s l} = b^{n-1}.$$

We say that n *passes the strong pseudoprime test at base* b if either $b^l \equiv 1(\bmod n)$ or $b^{2^r l} \equiv -1(\bmod n)$ for some $0 \leqslant r < s$.

Note that in the sequence $b^l, b^{2l}, b^{4l}, \ldots, b^{2^{s-1}l}, b^{2^s l}$, each element is the square of the preceding element. Thus if one of these elements is 1 or -1, then all the following elements are equal to 1.

Remark 3.1. If n passes the strong pseudoprime test at base b, then it also passes the Fermat pseudoprime test at base b. In particular, any strong pseudoprime is a Fermat pseudoprime. Proof: If n passes the strong pseudoprime test at b, then either $b^l \equiv 1(\bmod n)$ or $b^{2^r l} \equiv -1(\bmod n)$ for some $r < s$. In either case, $b^{2^s l} \equiv 1(\bmod n)$, and hence $b^{n-1} \equiv 1(\bmod n)$.

Remark 3.2. Any prime is a strong pseudoprime. Proof: If n is prime, then $\mathbb{Z}_n$ is a field. It follows that the polynomial $x^2 - 1$ has at most two roots in $\mathbb{Z}_n$. These roots are ± 1. By Fermat's Little Theorem, $b^{2^s l} = b^{n-1} = 1(\bmod n)$. If $b^l \neq 1(\bmod n)$, then let r be maximal such that $b^{2^r l} \neq 1$. Then $(b^{2^r l})^2 = 1$ implies $b^{2^r l} = -1$, so n passes the test at b.

Proposition 3.3. *If n is not prime, then n fails the strong pseudoprime test at base b for at least half of the elements $b \in \{1, \ldots, n-1\}$.*

Proof. Let $n - 1 = 2^s l$ as before. If n is not a Fermat pseudoprime, then the result follows from Proposition 1.2 and Remark 3.1. So let us consider the case where n is a Carmichael number. By Proposition 2.2, we can write $n = m_1 m_2$, where $m_1, m_2 \geqslant 3$ and $\gcd(m_1, m_2) = 1$. Since l is odd, we have $(-1)^l \not\equiv 1(\bmod n)$. Let r be the maximal integer such that there exists some $b \in \mathbb{Z}_n^*$ with $b^{2^r l} \not\equiv 1(\bmod n)$. Note that $0 \leqslant r < s$. Let

$$G = \{b \in \mathbb{Z}_n^* \mid b^{2^r l} \equiv \pm 1(\bmod n)\}.$$

Clearly, G is a subgroup of $\mathbb{Z}_n^*$, hence $|G|$ divides $|Z_n^*|$. We now show that G is a strict subset of $\mathbb{Z}_n^*$. By definition of r, there exists some $b \in \mathbb{Z}_n^*$ with $b^{2^r l} \not\equiv 1(\bmod n)$. Then either $b \notin G$, or else $b^{2^r l} \equiv -1(\bmod n)$. In the latter case, use the Chinese Remainder Theorem to define $b' \in \mathbb{Z}_n^*$ such that $b' \equiv b(\bmod m_1)$ and $b' \equiv 1(\bmod m_2)$. Then $b'^{2^r l} \equiv -1(\bmod m_1)$ and $b'^{2^r l} \equiv 1(\bmod m_2)$. This implies $b'^{2^r l} \not\equiv \pm 1(\bmod n)$, hence $b' \notin G$. In either case, $G \neq \mathbb{Z}_n^*$. Thus, $|G| < |\mathbb{Z}_n^*|$, hence $|G| \leqslant \frac{1}{2}|\mathbb{Z}_n^*| \leqslant \frac{n-1}{2}$.

Finally, we claim that for all $b \in \{1, \ldots, n-1\}$ with $b \notin G$, n fails the strong pseudoprime test at base b. Indeed, either b is not a unit, in which case $b^{n-1} \not\equiv 1(\bmod n)$. Or else, $b^{2^{r+1} l} \equiv 1(\bmod n)$ but $b^{2^r l} \not\equiv \pm 1(\bmod n)$, causing the test to fail. As there are at least $\frac{n-1}{2}$ elements in $\{1, \ldots, n-1\} \setminus G$, we are done. □

As a result of Remark 3.2 and Proposition 3.3, we obtain an efficient probabilistic algorithm for primality testing. This algorithm is known as the Miller-Rabin algorithm. Notice that the algorithm is correct for all numbers; there is no equivalent of Carmichael numbers with respect to strong pseudoprimes. A number is a strong pseudoprime if and only if it is prime, which is the case if and only if it passes (with probability as close to 1 as desired) the Miller-Rabin primality test. We finish by summarizing the algorithm:

Algorithm 3.4 (Miller-Rabin primality test).

Input: Integers $n \geqslant 2$ and $t \geqslant 1$.

Output: If n is prime, output "yes". If n is not prime, output "no" with probability at least $1 - 1/2^t$, and "yes" with probability at most $1/2^t$.

Algorithm: Let $n - 1 = 2^s l$, where l is odd. Pick t independent, uniformly distributed random numbers $b_1, \ldots, b_t \in \{1, \ldots, n-1\}$. For each i, check that one of the following conditions hold: either $b_i^l \equiv 1(\bmod n)$ or $b_i^{2^r l} \equiv -1(\bmod n)$ for some $0 \leqslant r < s$. If this is the case for all b_i, output "yes", else "no". □