MAT 3361, INTRODUCTION TO MATHEMATICAL LOGIC, Fall 2004

Handout 1: Rules of Fitch-style natural deduction

Conjunction introduction ($\wedge$I)

$$\begin{array}{l|l}
\vdots & \vdots \\
m & A \\
\vdots & \vdots \\
n & B \\
\vdots & \vdots \\
p & A \wedge B \qquad \wedge\text{I}, m, n
\end{array}
\qquad
\begin{array}{l|l}
\vdots & \vdots \\
m & B \\
\vdots & \vdots \\
n & A \\
\vdots & \vdots \\
p & A \wedge B \qquad \wedge\text{I}, m, n
\end{array}$$

Conjunction elimination ($\wedge$E)

$$\begin{array}{l|l}
\vdots & \vdots \\
m & A \wedge B \\
\vdots & \vdots \\
n & A \qquad \wedge\text{E}, m
\end{array}
\qquad
\begin{array}{l|l}
\vdots & \vdots \\
m & A \wedge B \\
\vdots & \vdots \\
n & B \qquad \wedge\text{E}, m
\end{array}$$

Disjunction Introduction ($\vee$I)

$$\begin{array}{l|l}
\vdots & \vdots \\
m & A \\
\vdots & \vdots \\
n & A \vee B \qquad \vee\text{I}, m
\end{array}
\qquad
\begin{array}{l|l}
\vdots & \vdots \\
m & B \\
\vdots & \vdots \\
n & A \vee B \qquad \vee\text{I}, m
\end{array}$$

Disjunction Elimination ($\vee$E)

$$\begin{array}{l|l}
\vdots & \vdots \\
m & A \vee B \\
m+1 & \quad \vert\; \underline{A} \\
\vdots & \quad \vert\; \vdots \\
n & \quad \vert\; \varphi \\
n+1 & \quad \vert\; \underline{B} \\
\vdots & \quad \vert\; \vdots \\
p & \quad \vert\; \varphi \\
p+1 & \varphi \qquad \vee\text{E}, m, (m+1)\text{–}n, (n+1)\text{–}p
\end{array}$$

Implication Introduction ($\Rightarrow$I)

$$\begin{array}{l|l}
\vdots & \vdots \\
m & \quad \vert\; \underline{A} \\
\vdots & \quad \vert\; \vdots \\
n & \quad \vert\; B \\
n+1 & A \Rightarrow B \qquad \Rightarrow\text{I}, m\text{–}n
\end{array}$$

Implication Elimination ($\Rightarrow$E)

$$\begin{array}{l|l}
\vdots & \vdots \\
m & A \\
\vdots & \vdots \\
n & A \Rightarrow B \\
\vdots & \vdots \\
p & B \qquad \Rightarrow\text{E}, m, n
\end{array}
\qquad
\begin{array}{l|l}
\vdots & \vdots \\
m & A \Rightarrow B \\
\vdots & \vdots \\
n & A \\
\vdots & \vdots \\
p & B \qquad \Rightarrow\text{E}, m, n
\end{array}$$

Negation Introduction ($\neg$I)

$$\begin{array}{l|l}
\vdots & \vdots \\
m & \quad \vert\; \underline{A} \\
\vdots & \quad \vert\; \vdots \\
n & \quad \vert\; \bot \\
n+1 & \neg A \qquad \neg\text{I}, m\text{–}n
\end{array}$$

Negation Elimination ($\neg$E)

$$\begin{array}{l|l}
\vdots & \vdots \\
m & A \\
\vdots & \vdots \\
n & \neg A \\
\vdots & \vdots \\
p & \bot \qquad \neg\text{E}, m, n
\end{array}
\qquad
\begin{array}{l|l}
\vdots & \vdots \\
m & \neg A \\
\vdots & \vdots \\
n & A \\
\vdots & \vdots \\
p & \bot \qquad \neg\text{E}, m, n
\end{array}$$

Contradiction Elimination ($\bot$E)

$$\begin{array}{l|l}
\vdots & \vdots \\
m & \bot \\
\vdots & \vdots \\
n & C \qquad \bot\text{E}, m
\end{array}$$

Double negation elimination ($\neg\neg$E)

$$\begin{array}{l|l}
\vdots & \vdots \\
m & \neg\neg A \\
\vdots & \vdots \\
n & A \qquad \neg\neg\text{E}, m
\end{array}$$

Repetition (R)

$$\begin{array}{l|l}
m & A \\
\vdots & \vdots \\
\vdots & \cdots \vert\; \vdots \\
n & \quad \vert\; A \qquad \text{R}, m
\end{array}$$

Forall-introduction ($\forall$I)

$$\begin{array}{l|l}
\vdots & \vdots \\
m & u \;\vert\; \vdots \\
\vdots & \quad \vert\; \vdots \\
n & \quad \vert\; A(u) \\
n+1 & \forall x\, A(x) \qquad \forall\text{I}, m\text{–}n
\end{array}$$

Forall-elimination ($\forall$E)

$$\begin{array}{l|l}
\vdots & \vdots \\
m & \forall x\, A(x) \\
\vdots & \vdots \\
n & A(t) \qquad \forall\text{E}, m
\end{array}$$

Exists-Introduction ($\exists$I)

$$\begin{array}{l|l}
\vdots & \vdots \\
m & A(t) \\
\vdots & \vdots \\
n & \exists x\, A(x) \qquad \exists\text{I}, m
\end{array}$$

Exists-Elimination ($\exists$E)

$$\begin{array}{l|l}
\vdots & \vdots \\
p & \exists x\, A(x) \\
m & u \;\vert\; \underline{A(u)} \\
\vdots & \quad \vert\; \vdots \\
n & \quad \vert\; \varphi \\
n+1 & \varphi \qquad \exists\text{E}, p, m\text{–}n
\end{array}$$

The biconditional

To simplify our formal proof system, we do not introduce any special rules for the connective $\Leftrightarrow$. Instead, we simply regard the formula $A \Leftrightarrow B$ as an *abbreviation* for $(A \Rightarrow B) \wedge (B \Rightarrow A)$.

Repetition (R)

Let A be a formula written at line k (either as a hypothesis, or as a formula already proven). Then one can repeat A at line m if:

(1) $k < m$, and

(2) every vertical from line k continues without interruption to line m.

Examples:

This is ok:

k	$\vert\ \underline{A}$	
$\vdots$	$\vert\ \vdots$	
m	$\vert\ A$	R, k

This too:

$\vdots$	$\vert\ \vdots$	
k	$\vert\ A$	
$\vdots$	$\vert\ \vdots$	
n	$\vert\quad \vert\ \underline{B}$	
$\vdots$	$\vert\quad \vert\ \vdots$	
m	$\vert\quad \vert\ A$	R, k

But not this:

$\vdots$	$\vert\quad \vert\ \vdots$	
k	$\vert\quad \vert\ A$	
$\vdots$	$\vert\quad \vert\ \vdots$	
n	$\vert\quad \vert\ \underline{B}$	
$\vdots$	$\vert\quad \vert\ \vdots$	
m	$\vert\quad \vert\ A$	R, k

Derived rules

It is possible to create additional rules to be used in natural deduction proofs. These rules are derived from the official rules which are stated above; they can be regarded as "shortcuts". If you want to use such a derived rule, you first have to prove it (i.e., give a separate formal proof which justifies the rule).

One example of a derived rule is De Morgan's law for disjunction,

$$\neg(A \vee B) \vdash \neg A \wedge \neg B.$$

We can give a formal proof of De Morgan's law using only the rules of natural deduction given above:

1	$\vert\ \underline{\neg(A \vee B)}$	
2	$\vert\quad \vert\ \underline{A}$	
3	$\vert\quad \vert\ A \vee B$	$\vee$I, 2
4	$\vert\quad \vert\ \neg(A \vee B)$	R, 1
5	$\vert\quad \vert\ \bot$	$\neg$E, 3, 4
6	$\vert\ \neg A$	$\neg$I, 2–5
7	$\vert\quad \vert\ \underline{B}$	
8	$\vert\quad \vert\ A \vee B$	$\vee$I, 7
9	$\vert\quad \vert\ \neg(A \vee B)$	R, 1
10	$\vert\quad \vert\ \bot$	$\neg$E, 8, 9
11	$\vert\ \neg B$	$\neg$I, 7–10
12	$\vert\ \neg A \wedge \neg B$	$\wedge$I, 6, 11

Having given this formal proof, we can now use De Morgan's law as a *derived rule*, as follows:

$$\begin{array}{|l|ll|}\hline \vdots & \vdots & \\ m & \neg(A \vee B) & \\ \vdots & \vdots & \\ n & \neg A \wedge \neg B & \text{De Morgan, } m \\ \hline \end{array}$$

Note that there are three other De Morgan's laws, namely

$$\begin{aligned} \neg A \wedge \neg B &\vdash \neg(A \vee B) \\ \neg(A \wedge B) &\vdash \neg A \vee \neg B \\ \neg A \vee \neg B &\vdash \neg(A \wedge B) \end{aligned}$$

Each of them must be proven separately in natural deduction; thereafter, it can be used as a derived rule.

Problem 1. Give formal proofs of the remaining three laws of De Morgan.

Problem 2. For any proposition φ, let $r(\varphi)$ be the rank of φ and let $c(\varphi)$ be the number of connectives in φ (connectives are $\{\perp, \wedge, \vee, \rightarrow, \leftrightarrow, \neg\}$). (a) Write down recursive definitions of r and c (for r, a definition was already given in class). (b) Prove, by induction, that $r(\varphi) \leqslant c(\varphi)$ for all $\varphi \in \mathrm{PROP}$.

Problem 3. Prove that there exists no $\varphi \in \mathrm{PROP}$ such that the length of φ is 6 symbols.

Problem 4. For the purpose of this problem, we ignore the connectives "$\perp$", "$\rightarrow$" and "$\leftrightarrow$", i.e., we consider propositions built from "$\wedge$", "$\vee$", and "$\neg$" only. The *De Morgan dual* of a proposition φ is defined as follows:

$$\begin{aligned} &dm(p_i) = p_i \\ &dm(\,(\varphi \wedge \psi)\,) = (dm(\varphi) \vee dm(\psi)) \\ &dm(\,(\varphi \vee \psi)\,) = (dm(\varphi) \wedge dm(\psi)) \\ &dm(\,(\neg \varphi)\,) = (\neg\, dm(\varphi)) \end{aligned}$$

(a) Let r be the rank function. Prove $r(\varphi) = r(dm(\varphi))$ for all φ.

(b) Let $[\![-]\!]$ be a valuation, and define $[\![-]\!]'$ by $[\![\varphi]\!]' = 1 - [\![dm(\varphi)]\!]$, for all φ. Prove that $[\![-]\!]'$ is a valuation.

(c) A proposition φ is called *satisfiable* if there exists a valuation $[\![-]\!]$ such that $[\![\varphi]\!] = 1$. Prove that φ is satisfiable if and only if $dm(\varphi)$ is not valid.

9. $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$.

10. $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$.

11. $A \wedge (A \vee B) \equiv A$.

12. $A \vee (A \wedge B) \equiv A$.

13. $\vdash \neg(A \wedge (\neg A))$.

14. $A \rightarrow (\neg A) \vdash \neg A$.

15. $(A \rightarrow B) \wedge (A \rightarrow C) \equiv A \rightarrow (B \wedge C)$.

16. $A \rightarrow B,\ B \rightarrow C \vdash A \rightarrow C$.

17. $\vdash A \rightarrow (B \rightarrow A)$.

18. $(A \rightarrow B) \rightarrow C \vdash B \rightarrow C$.

19. $A \rightarrow B \vdash (\neg B) \rightarrow (\neg A)$.

20. $A \rightarrow B \vdash \neg(A \wedge (\neg B))$.

21. $A \wedge (\neg B) \vdash \neg(A \wedge B)$.

22. $(\neg A) \rightarrow (\neg B) \vdash B \rightarrow (\neg(A \rightarrow (\neg B)))$.

23. $\vdash \neg(A \leftrightarrow (\neg A))$.

24. $A \vee B \vdash (B \rightarrow A) \rightarrow A$.

25. $A \vee B \vdash (\neg B) \rightarrow (C \rightarrow A)$.

26. $(B \rightarrow A) \wedge (A \vee B) \vdash A$.

27. $A \vee B \vdash (\neg A) \rightarrow B$.

28. $(\neg A) \vee B \vdash A \rightarrow B$.

29. $(\neg A) \vee (\neg B) \vdash \neg(A \wedge B)$.

30. $\neg(A \vee B) \equiv (\neg A) \wedge (\neg B)$.

31. $(A \vee B) \rightarrow C \equiv (A \rightarrow C) \wedge (B \rightarrow C)$.

32. $(A \rightarrow B) \vee (A \rightarrow C) \vdash A \rightarrow (B \vee C)$.

33. $(A \rightarrow C) \vee (B \rightarrow C) \vdash (A \wedge B) \rightarrow C$.

34. $((\neg A) \vee C) \wedge (B \rightarrow C) \vdash (A \rightarrow B) \rightarrow C$.

35. $\vdash (A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$.

36. $(A \rightarrow C) \vee (B \rightarrow D) \vdash (A \wedge B) \rightarrow (C \vee D)$.

37. $A \rightarrow B,\ C \rightarrow D,\ (\neg B) \vee (\neg D) \vdash (\neg A) \vee (\neg C)$.

38. $A \rightarrow (C \vee D),\ (A \vee D) \vee E,\ A \rightarrow (\neg C) \vdash D \vee E$.

39. $A \vdash \neg\neg A$.

40. $\neg\neg\neg A \equiv \neg A$.

(NOTE: The $\neg\neg$ rule is required for nos. 41-53.)

41*. $\vdash A \vee (\neg A)$.

42. $(A \rightarrow B) \vee C,\ A \rightarrow (\neg C) \vdash (B \rightarrow C) \rightarrow (\neg A)$.

43. $\neg(A \wedge (\neg B)) \vdash A \rightarrow B$.

44. $(\neg B) \rightarrow (\neg A) \vdash A \rightarrow B$.

45*. $A \rightarrow B \vdash (\neg A) \vee B$.

46*. $(B \rightarrow A) \rightarrow A \vdash A \vee B$.

47*. $(A \rightarrow B) \rightarrow C \vdash A \vee C$.

48*. $(\neg A) \rightarrow B \vdash A \vee ((\neg A) \wedge B)$.

49*. $\vdash (A \wedge B) \vee (\neg A) \vee (\neg B)$.

50*. $\neg(A \wedge B) \vdash (\neg A) \vee (\neg B)$.

51*. $\vdash (A \rightarrow B) \vee (B \rightarrow A)$

52*. $A \rightarrow (B \vee C) \vdash (A \rightarrow B) \vee (A \rightarrow C)$.

53*. $(A \wedge B) \rightarrow (C \vee D) \vdash (A \rightarrow C) \vee (B \rightarrow D)$. (cf.no.36)

54. Prove that if $\varphi \wedge \psi \vdash \theta$ and $\varphi \wedge \theta \vdash \zeta$ then $\varphi \wedge \psi \vdash \zeta$.

55. Prove that if $\varphi \vdash \psi$ then

(i) $\varphi \wedge \theta \vdash \psi \wedge \theta$,
(ii) $\varphi \vee \theta \vdash \psi \vee \theta$,
(iii) $\theta \rightarrow \varphi \vdash \theta \rightarrow \psi$,
(iv) $\psi \rightarrow \theta \vdash \varphi \rightarrow \theta$.

56. Prove that $\Gamma \cup \{\varphi\} \vdash \psi$ iff $\Gamma \vdash \varphi \rightarrow \psi$ (Γ is any set of formulae).