MAT 5361, TOPICS IN QUANTUM COMPUTATION, WINTER 2004
Lecture Notes 1: A Hahn-Banach style theorem for normed cones

1 Abstract cones

Let $\mathbb{R}_+$ be the set of non-negative reals. An *abstract cone* is analogous to a real vector space, except that we take the non-negative reals as scalars. Since the non-negative reals do not form a field, we have to replace the vector space law $v + (-v) = 0$ by a *cancelation law* $v + u = w + u \Rightarrow v = w$. We also require *strictness*, which means, no non-zero element has a negative.

Definition (Abstract cone). An ***abstract cone*** is a set C, together with two operations $+ : C \times C \to C$ and $\cdot : \mathbb{R}_+ \times C \to C$ and a distinguished element $0 \in C$, satisfying the following laws for all $v, w, u \in C$ and $\lambda, \mu \in \mathbb{R}_+$:

Additive laws:
$0 + v = v$
$v + (w + u) = (v + w) + u$
$v + w = w + v$
$v + u = w + u \quad \Rightarrow \quad v = w$ ***(cancelation)***
$v + w = 0 \quad \Rightarrow \quad v = w = 0$ ***(strictness)***

Multiplicative laws:
$1v = v$
$(\lambda\mu)v = \lambda(\mu v)$
$(\lambda + \mu)v = \lambda v + \mu v$
$\lambda(v + w) = \lambda v + \lambda w,$

Examples. $\mathbb{R}_+$ is an abstract cone. The set

$$\mathbb{R}_+^n = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{R}_+\}$$

is an abstract cone, with the coordinatewise operations. More generally, if $C_1, \ldots, C_n$ are abstract cones, then so is $C_1 \times \ldots \times C_n$. The set of all complex hermitian positive $n \times n$-matrices,

$$\mathcal{P}_n = \{A \in \mathbb{C}^{n \times n} \mid A = A^* \text{ and } \forall v \in \mathbb{C}^n . v^* A v \geqslant 0\}$$

is an abstract cone. Also, for any signature $\sigma = n_1, \ldots, n_s$, the set of positive matrix tuples $\mathcal{P}_\sigma := \mathcal{P}_{n_1} \times \ldots \times \mathcal{P}_{n_s}$ is an abstract cone.

Definition (Linear function of abstract cones). A ***linear function*** of abstract cones is a function $f : C \to D$ such that $f(v + w) = f(v) + f(w)$ and $f(\lambda v) = \lambda f(v)$, for all $v, w \in C$ and $\lambda \in \mathbb{R}_+$.

Remark. Every abstract cone C can be completed to a real vector space V_C. The elements of V_C are pairs (v, w), where $v, w \in C$, modulo the equivalence relation $(v, w) \sim (v', w')$ if $v + w' = v' + w$. Addition and multiplication by non-negative scalars are defined pointwise, and we define $-(v, w) = (w, v)$. Moreover, any linear function of abstract cones $f : C \to C'$ extends uniquely to a linear function of vector spaces $f : V_C \to V_{C'}$.

We say that an abstract cone C is ***finite dimensional*** if V_C is a finite dimensional vector space. Note that, unlike vector spaces, finite dimensional cones need not be spanned by a finite set. A counterexample is $C = \{(x, y, z) \in \mathbb{R}^3 \mid \sqrt{x^2 + y^2} \leqslant z\}$.

Definition (Convexity). A subset D of an abstract cone C is said to be ***convex*** if for all $u, v \in D$ and $\lambda \in [0, 1]$, $\lambda u + (1 - \lambda)v \in D$. The ***convex closure*** of a set D is defined to be the smallest convex set containing D.

2 The cone order

Definition (Cone order). Let C be an abstract cone. The ***cone order*** is defined by setting $v \sqsubseteq w$ if there exists $u \in C$ such that $v + u = w$. Note that the cone order is a partial order. If $v \sqsubseteq w$, then the element u such that $v + u = w$ is necessarily unique, and thus we may also write $u = w - v$.

Remark. Note that every linear function of abstract cones $f : C \to D$ is also *monotone*, i.e., $v \sqsubseteq v'$ implies $f(v) \sqsubseteq f(v')$.

Examples. On $\mathbb{R}_+$, the cone order is just the usual order $\leqslant$ of the reals. On $\mathbb{R}_+^n$, it is the pointwise order. On $\mathcal{P}_\sigma$, it is the so-called *Löwner partial order*.

Definition (Down closure). Let $D \subseteq C$ be a subset of an abstract cone. We define its ***down-closure*** $\downarrow D$ to be the set $\{u \in C | \exists v \in D . u \sqsubseteq v\}$. We say that D is ***down-closed*** if $D = \downarrow D$. The concept of ***up-closure*** is defined dually.

Lemma 2.1. *The down-closure of a convex set is convex.*

Proof. We use the easily verified fact that addition and scalar multiplication are monotone, thus $u' \sqsubseteq u$ and $v' \sqsubseteq v$ implies $\lambda u' + (1 - \lambda)v' \sqsubseteq \lambda u + (1 - \lambda)v$. □

3 A separation theorem for abstract cones

Definition (Generating set). Let C be an abstract cone, and let $D \subseteq C$ be a down-closed, convex set. We say that D ***generates C*** if for all $v \in C$, there exists some $\lambda > 0$ such that $\lambda v \in D$.

Theorem 3.1 (Separation). *Let C be an abstract cone, let U and D be convex sets such that U is up-closed, D is down-closed, and $U \cap D = \emptyset$. Moreover, assume that D generates C. Then there exists a linear function $f : C \to \mathbb{R}_+$ such that $f(v) \leqslant 1$ for all $v \in D$ and $f(u) \geqslant 1$ for all $u \in U$.*

Proof. Let $\mathcal{E}$ be the class of subsets $E \subseteq C$ with the following properties: E is convex and down-closed, $D \subseteq E$, and $E \cap U = \emptyset$. Clearly $D \in \mathcal{E}$, and moreover, $\mathcal{E}$ is closed under unions of chains; therefore, by Zorn's Lemma, there is a maximal element in $\mathcal{E}$ with respect to inclusion.

Let E be maximal in $\mathcal{E}$, and let $E^c = C \setminus E$ be its complement. We will prove that E^c is convex. We use the following convention: for scalars $\lambda \in [0, 1]$, we write $\overline{\lambda} = 1 - \lambda$. We first claim that for every $v \in E^c$, the convex closure of $E \cup \{v\}$ intersects U. Namely, let E_v be this convex closure. Then $\downarrow E_v$ is convex by Lemma 2.1. By maximality of E, we must have $\downarrow E_v \cap U \neq \emptyset$, and therefore $E_v \cap U \neq \emptyset$ since U is up-closed.

Now assume that E^c is not convex. Then there exist $v_0, v_1 \in E^c$ and $\lambda \in [0,1]$ such that $\lambda v_0 + \overline{\lambda} v_1 \in E$. By the previous paragraph, for $i = 0, 1$, we can find $e_i \in E$ and $\mu_i \in [0,1]$ such that $\mu_i v_i + \overline{\mu}_i e_i \in U$. Note that $e_i \notin U$ implies $\mu_i \neq 0$. Let $w = \lambda v_0 + \overline{\lambda} v_1$ and $u_i = \mu_i v_i + \overline{\mu}_i e_i$. Then we have:

$$\frac{\lambda\mu_1}{\lambda\mu_1 + \overline{\lambda}\mu_0} u_0 + \frac{\overline{\lambda}\mu_0}{\lambda\mu_1 + \overline{\lambda}\mu_0} u_1 = \frac{\lambda\mu_1\overline{\mu_0}}{\lambda\mu_1 + \overline{\lambda}\mu_0} e_0 + \frac{\overline{\lambda}\,\overline{\mu_1}\mu_0}{\lambda\mu_1 + \overline{\lambda}\mu_0} e_1 + \frac{\mu_1\mu_0}{\lambda\mu_1 + \overline{\lambda}\mu_0} w.$$

The left-hand-side is a convex combination of $u_0, u_1 \in U$, and the right-hand-side is a convex combination of $e_0, e_1, w \in E$. This contradicts the fact that U and E are convex and disjoint, proving that E^c is convex.

If A is a subset of a cone, we write $\lambda A = \{\lambda a \mid a \in A\}$. Note that A is convex iff for all $\lambda, \mu \geqslant 0$, $\lambda A + \mu A \subseteq (\lambda + \mu)A$. We now define the function f by

$$f(v) = \inf\{\lambda \mid v \in \lambda E,\ \lambda > 0\}.$$

Note that because $D \subseteq E$ and D generates C, the set E also generates C. Therefore, for all $v \in C$, there exists some λ such that $v \in \lambda E$. Thus, $f(v)$ is well-defined and finite. Moreover, since $D \subseteq E$, it follows that $f(v) \leqslant 1$ for all $v \in D$. On the other hand, if $u \in U$, then for all $\lambda \leqslant 1$, $u \notin \lambda E$; thus $f(u) \geqslant 1$. It remains to be shown that f is linear.

First, we show that f is monotone; this follows directly from the definition and the fact that E is down-closed. Also immediate is the fact that $f(\lambda v) = \lambda f(v)$. The inequality $f(v+w) \leqslant f(v) + f(w)$ follows from the convexity of E.

To prove the opposite inequality $f(v) + f(w) \leqslant f(v+w)$, we consider two cases. If $f(v) = 0$ or $f(w) = 0$, then this inequality follows from monotonicity. Otherwise, suppose $f(v), f(w) \neq 0$. Consider any $\lambda, \mu > 0$ such that $\lambda < f(v)$ and $\mu < f(w)$. Then by definition of f, we have $v \notin \lambda E$ and $w \notin \mu E$, hence $v \in \lambda E^c$ and $w \in \mu E^c$. Convexity of E^c implies that $v + w \in (\lambda + \mu)E^c$, hence $\lambda + \mu \leqslant f(v+w)$. Since λ, μ were arbitrary, this shows $f(v) + f(w) \leqslant f(v+w)$. □

4 Normed cones

Definition (Normed cone). Let C be an abstract cone. A ***norm*** on C is a function $\|-\| : C \to \mathbb{R}_+$ satisfying the following conditions for all $v, w \in C$ and $\lambda \in \mathbb{R}_+$:

$\|v + w\| \leqslant \|v\| + \|w\|$	***(triangle inequality)***
$\|\lambda v\| = \lambda \|v\|$	***(linearity)***
$\|v\| = 0 \Rightarrow v = 0$	***(strictness)***
$v \sqsubseteq w \Rightarrow \|v\| \leqslant \|w\|$	***(monotonicity)***

A ***normed cone*** $\mathbf{C} = \langle C, \|-\|_{\mathbf{C}} \rangle$ is an abstract cone C equipped with a norm $\|-\|_{\mathbf{C}}$.

The first three conditions are just the usual conditions for a norm on a vector space, except of course that the scalar property is restricted to non-negative scalars. The last condition ensures that the norm is *monotone*. Note that monotonicity does not follow from the remaining three axioms.

If $\mathbf{C} = \langle C, \|-\|_{\mathbf{C}} \rangle$ is a normed cone, we define its ***unit ideal*** to be the set

$$\mathcal{D}_{\mathbf{C}} = \{v \in C \mid \|v\|_{\mathbf{C}} \leqslant 1\}$$

The unit ideal is a down-closed and convex subset of C.

Definition (Non-expanding linear function). Let $\mathbf{C}$ and $\mathbf{C}'$ be normed cones. A linear function $f : \mathbf{C} \to \mathbf{C}'$ is ***non-expanding*** (or ***norm non-increasing***) if for all $v \in C$, $\|f(v)\|_{\mathbf{C}'} \leqslant \|v\|_{\mathbf{C}}$.

5 A Hahn-Banach style theorem for normed cones

Theorem 5.1. *Let $\mathbf{C}$ be a normed cone, and let $u \in C$ with $\|u\|_{\mathbf{C}} = 1$. Then there exists a non-expanding linear function $f : C \to \mathbb{R}_+$ such that $f(u) = 1$.*

Proof. Apply Theorem 3.1 to the sets $D = \mathcal{D}_{\mathbf{C}}$ and $U = \uparrow\{\lambda u \mid \lambda > 1\}$. □