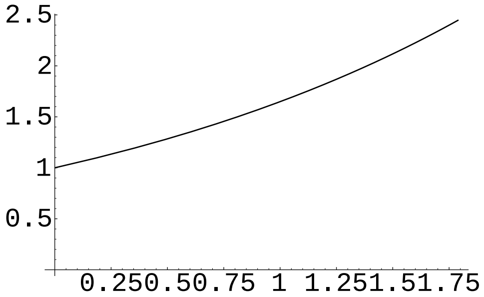
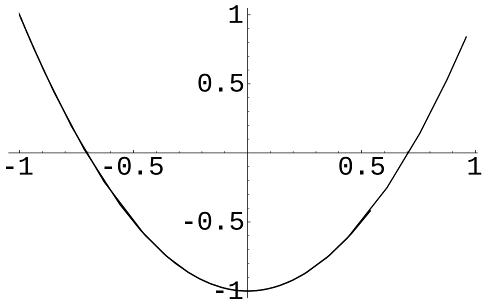
MATH 285, HONORS MULTIVARIABLE CALCULUS, FALL 1999

Answers to Homework Set 1

Problem 11.1.12 (a) From $x = \ln t$, we get $t = e^x$, and thus $y = \sqrt{e^x} = e^{x/2}$. (b) The curve is traced from left to right.



Problem 11.1.14 (a) We can write $y = \cos 2t = 2\cos^2 t - 1 = 2x^2 - 1$. (b) The curve is traced back and forth periodically as the parameter increases. Notice that the range of x is $-1 \leqslant x \leqslant 1$.



Problem 11.1.36 First, by looking at the small circle, we see that the y-coordinate of P is given by $y = b\sin\theta$. The x-coordinate of P is the same as the x-coordinate of B. We have

$$\frac{x}{a} = \frac{|OA|}{|OB|} = \cos\theta,$$

and thus $x = a/\cos\theta$.

Problem 11.1.38 Suppose we take the angle between the line OB and the x-axis to be our parameter θ. Call the point where the red line intersects the x-axis C. By looking at the triangle OBC, we see that $|CB| = 2a\tan\theta$. Then, by looking at the triangle CBA, we see that $|AB| = |CB|\sin\theta = 2a\sin\theta\tan\theta = 2a\sin^2\theta/\cos\theta$. Since $|OP| = |AB|$, we can now figure out the x- and y-coordinates of P:

$$x = |OP|\cos\theta = 2a\sin^2\theta$$
$$y = |OP|\sin\theta = 2a\sin^3\theta/\cos\theta$$

Problem 11.2.8 First, we calculate the slope of the curve at the given point:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t - t\sin t}{\sin t + t\cos t},$$

which equals $\frac{-1-0}{0-\pi} = \frac{1}{\pi}$ at the point $t = \pi$. Also, for $t = \pi$, we get $(x, y) = (0, -\pi)$. So we need the equation of a line of slope $1/\pi$ through the point $(0, -\pi)$. The general equation of a line through (x_0, y_0) with slope m is

$$(y - y_0) = m(x - x_0),$$

so the equation of the desired tangent line is $(y + \pi) = (x - 0)/\pi$, or $y = x/\pi - \pi$.

Problem 11.2.9 We calculate the slope by two methods: (a) The point in question is $t = 0$, where $(x, y) = (1, 1)$. The slope is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2(t-1)}{e^t},$$

which equals -2 at the point $t = 0$. (b) By first eliminating t: we get $t = \ln x$, and thus $y = (\ln x - 1)^2$. The slope is

$$\frac{dy}{dx} = 2(\ln x - 1)\cdot\frac{1}{x},$$

which equals (no surprise!) -2 when $x = 1$. Thus, the equation of the tangent line is $(y - 1) = -2(x - 1)$, or $y = -2x + 3$.

Problem 11.2.22 First, notice that the parameter θ is cyclic with period 2π, so that we need to concentrate only on points in the interval $\theta \in [-\pi, \pi]$. We first set $dx/d\theta$ equal to zero:

$$\frac{dx}{d\theta} = a(-\sin\theta + 2\cos\theta\sin\theta) = a\sin\theta(2\cos\theta - 1) = 0.$$

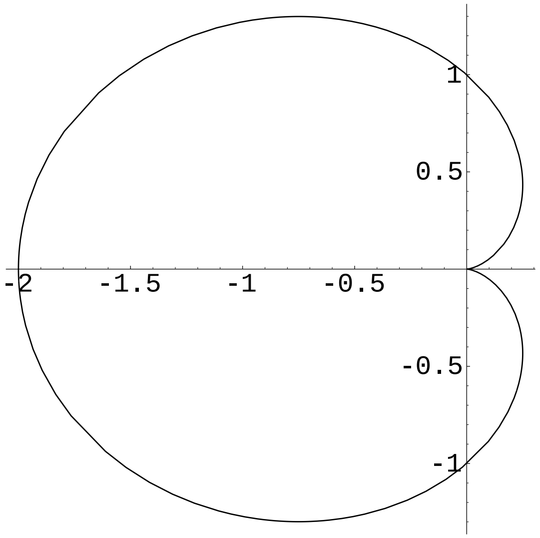
This happens either when $\sin\theta = 0$, i.e., when $\theta \in \{0, \pi, -\pi\}$, or when $2\cos\theta - 1 = 0$, i.e., $\cos\theta = 1/2$, i.e., when $\theta \in \{\pi/3, -\pi/3\}$. Now we set $dy/d\theta = 0$:

$$\begin{aligned}\frac{dy}{d\theta} &= a(\cos\theta - \cos^2\theta + \sin^2\theta) \\ &= a(\cos\theta + 1 - 2\cos^2\theta) \\ &= a(1-\cos\theta)(1+2\cos\theta) = 0.\end{aligned}$$

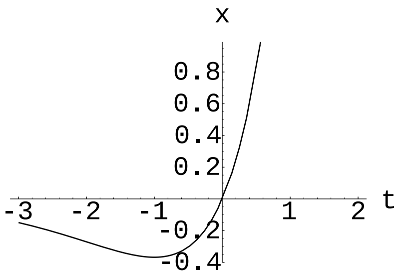
As you can see, this is the case if either $\cos\theta = 1$, i.e., at $\theta = 0$, or else $\cos\theta = -1/2$, i.e., when $\theta \in \{2\pi/3, -2\pi/3\}$. Thus, we have the following data:

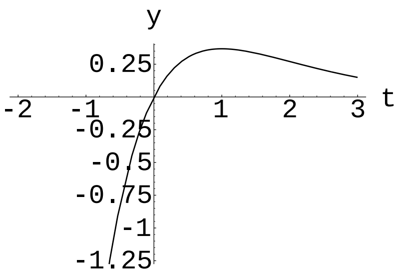
θ	$-2\pi/3$	$-\pi/3$	0	$\pi/3$	$2\pi/3$	$\pm\pi$
(x,y)	$(-\frac{3a}{4}, -\frac{3a\sqrt{3}}{4})$	$(\frac{a}{4}, -\frac{a\sqrt{3}}{4})$	$(0,0)$	$(\frac{a}{4}, \frac{a\sqrt{3}}{4})$	$(-\frac{3a}{4}, \frac{3a\sqrt{3}}{4})$	$(-2a, 0)$
dx/dt	$\neq 0$	$= 0$	$= 0$	$= 0$	$\neq 0$	$= 0$
dy/dt	$= 0$	$\neq 0$	$= 0$	$\neq 0$	$= 0$	$\neq 0$
tangent	horizontal	vertical	cusp	vertical	horizontal	vertical

As *cusp* is a place where the curve rests for an instant as it is traced out. Often, the curve will change directions at such a point, and it may have more than one tangent there. The graph looks as follows:

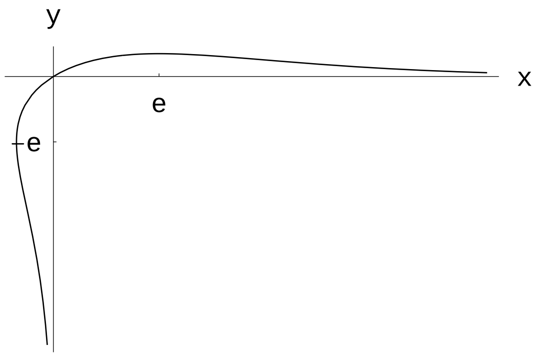


Problem 11.2.24 To estimate the highest point and the leftmost point, we first plot x and y separately as functions of t:





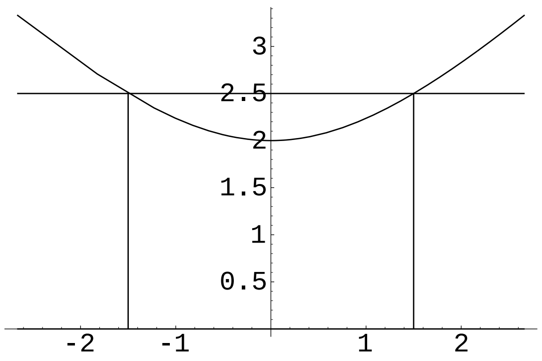
The highest point (maximum y coordinate) and leftmost point (minimum x coordinate) seem to occur near $t = 1$ and $t = -1$, respectively. A quick calculation shows that indeed $dy/dt = (1-t)e^{-t} = 0$ if $t = 1$, and $dx/dt = (1+t)e^{-t} = 0$ if $t = -1$. The parametric curve, in a slightly exaggerated graph, looks as follows:



Notice that the maximum y-value occurs at $t = 1$ or $x = e$. Similarly, the minimum x-value occurs at $t = -1$ or $y = -e$.

To determine the asymptotes of the curve, notice that as $t \to \infty$, we have $x \to \infty$ and $y \to 0$. Thus the x-axis is an asymptote for $t \to \infty$. Also, when $t \to -\infty$, we have $x \to 0$ and $y \to -\infty$, thus the y-axis is an asymptote for $t \to -\infty$.

Problem 11.2.34 We first sketch the given curve and the given horizontal line:



The find the enclosed area, we first determine the appropriate range for t. The given curve intersects the horizontal line when $y = t + 1/t = 2.5$, which happens at $t_0 = 2$ and $t_1 = 1/2$. The x-values at these two points are $x_0 = -1.5$ and $x_1 = 1.5$. The area between the curve and the x-axis, for this range of t, is

$$\begin{aligned}
\int_{t_0}^{t_1} yx'\,dt &= \int_{1/2}^{2} (t + \frac{1}{t})(1 + \frac{1}{t^2})\,dt \\
&= \int_{1/2}^{2} t + \frac{2}{t} + \frac{1}{t^3}\,dt \\
&= \left[\frac{t^2}{2} + 2\ln t - \frac{1}{2t^2}\right]_{1/2}^{2} \\
&= 15/4 + 4\ln 2.
\end{aligned}$$

We must subtract this area from the rectangle bounded by $y = 2.5$, $y = 0$, $x = -1.5$ and $x = 1.5$. The area of that rectangle is $2.5 \times 3 = 15/2$. So the area that we were looking for is $15/2 - (15/4 + 4\ln 2) = 15/4 - 4\ln 2 \approx 0.977411$.

Problem 11.2.36 To find the area enclosed by the closed curve $x = a\cos^3\theta$, $y = a\sin^3\theta$, we first note that the period of the parameter θ is 2π. This is important because we need to be sure that our path of integration winds around the area exactly once, or otherwise we might be counting the area multiple times. The area is

$$\int_{\theta_0}^{\theta_1} yx'\,d\theta = \int_0^{2\pi} 3a^2\sin^4\theta\cos^2\theta\,d\theta.$$

Omitting the bounds and the constant, we calculate the antiderivative according to the formula in the back of the book — not a very satisfying method, but it works.

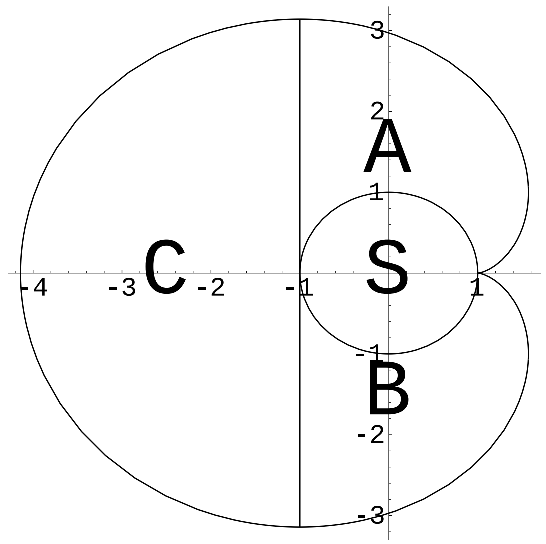
$$\begin{aligned}
\int \sin^4\theta\cos^2\theta\,d\theta &= \frac{\sin^5\theta\cos\theta}{6} + \frac{1}{6}\int \sin^4\theta\,d\theta \\
&= \frac{\sin^5\theta\cos\theta}{6} + \frac{1}{6}(-\frac{1}{4}\sin^3\theta\cos\theta + \frac{3}{4}\int \sin^2\theta\,d\theta) \\
&= \frac{\sin^5\theta\cos\theta}{6} + \frac{1}{6}(-\frac{1}{4}\sin^3\theta\cos\theta + \frac{3}{4}(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta)).
\end{aligned}$$

Notice that when we plug in the bounds 0 and 2π, all the terms containing θ disappear, and we are left with:

$$\begin{aligned}
&3a^2\left[\frac{\sin^5\theta\cos\theta}{6} + \frac{1}{6}(-\frac{1}{4}\sin^3\theta\cos\theta + \frac{3}{4}(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta))\right]_0^{2\pi} \\
&\qquad = 3a^2 \cdot \frac{1}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot 2\pi = \frac{3}{8}\pi a^2.
\end{aligned}$$

So the answer is precisely $3/8$ of the area of a circle of the same radius a.

Problem 11.2.42 We fix a coordinate system with the origin in the center of the silo, as shown here:



The tickmarks in this graph show multiples of r. The cow is attached at the point $(-r, 0)$ with a rope of length πr. The area where the cow lives consists of the three parts labeled A, B, and C. The silo is labeled S, and the cow cannot go there. The area A is bounded by the silo, the vertical line $x = -r$, and the parametric curve

$$\begin{aligned} x &= r(\cos\theta + \theta\sin\theta) \\ y &= r(\sin\theta - \theta\cos\theta), \text{ where } 0 \leqslant \theta \leqslant \pi. \end{aligned}$$

To justify these equations, consider the image in Problem 11.2.41 in the textbook. The coordinates of the point T are $(r\cos\theta, r\sin\theta)$. The distance $|TP|$ is $r\theta$. The coordinates of P relative to T are $(r\theta\sin\theta, -r\theta\cos\theta)$. Thus, the coordinates of P relative to O are the above parametric equations.

The area $A + S/2$ can be found by integrating the parametric curve:

$$\begin{aligned} \int_{\theta_0}^{\theta_1} yx'\,d\theta &= \int_0^\pi r(\sin\theta - \theta\cos\theta)\cdot r(-\sin\theta + \sin\theta + \theta\cos\theta)\,d\theta \\ &= r^2\int_0^\pi (\sin\theta - \theta\cos\theta)\theta\cos\theta\,d\theta \\ &= r^2\int_0^\pi \theta\sin\theta\cos\theta - \theta^2\cos^2\theta\,d\theta \\ &= r^2\int_0^\pi \frac{\theta}{2}\sin 2\theta - \frac{\theta^2}{2}(\cos 2\theta + 1)\,d\theta. \end{aligned}$$

We can separate this into two integrals, and integrate each one by parts. After a number of steps, we obtain

$$\frac{r^2}{12}\left[-2t^3 - 6t\cos 2t + 3\sin 2t - 3t^2\sin 2t\right]_0^\pi = -\frac{\pi}{2}r^2 - \frac{\pi^3}{6}r^2.$$

Note that the answer came out negative, because we traced out the curve from right to left, or counterclockwise. So the actual area of $A + S/2$ is $\pi r^2/2 + \pi^3 r^2/6$. Subtracting the area of half the silo, which is $\pi r^2/2$, we find that the area of A is $\pi^3 r^2/6$. The area B is, of course, the same, and the area C, which is just a semicircle, is $(\pi/2)(\pi r)^2 = \pi^3 r^2/2$. Thus, the total area available to the cow is:

$$A + B + C = \frac{5}{6}\pi^3 r^2$$