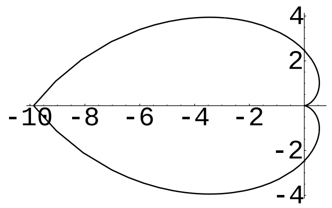
MATH 285, HONORS MULTIVARIABLE CALCULUS, FALL 1999

Answers to the First Midterm

Problem 1 (10 points) The curve shown on the right is given in polar coordinates by the equation $r = \theta^2$, where $-\pi \leqslant \theta \leqslant \pi$. Calculate the area enclosed by the curve.



The area of a curve in polar coordinates is

$$\int_{t_1}^{t_2} \frac{1}{2} r^2 \, d\theta = \int_{-\pi}^{\pi} \frac{1}{2} \theta^4 \, d\theta = \left[\frac{1}{2} \cdot \frac{\theta^5}{5} \right]_{-\pi}^{\pi} = \frac{\pi^5}{5}.$$

Problem 2 (8 points) Find an equation of the tangent line to the curve $x = e^t$, $y = \sin 2t$ at the point $t = 0$.

We first calculate the slope:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos 2t}{e^t} = \frac{2}{1} = 2.$$

The tangent line goes through the point $x_0 = e^0 = 1$, $y_0 = \sin 0 = 0$. The equation of the tangent line is $y - y_0 = 2(x - x_0)$ or $y = 2(x - 1)$.

Problem 3 (10 points) Find the curvature $\kappa(t)$ of the curve $\vec{\mathbf{r}} = \langle 3t^3 + t, t^2, 0 \rangle$ as a function of t. How many points along the curve are there at which $\kappa(t) = 0$?

We first calculate $\vec{\mathbf{r}}\,' = \langle 9t^2 + 1, 2t, 0 \rangle$ and $\vec{\mathbf{r}}\,'' = \langle 18t, 2, 0 \rangle$. We have $\vec{\mathbf{r}}\,' \times \vec{\mathbf{r}}\,'' = \langle 0, 0, (9t^2+1) \cdot 2 - 18t \cdot 2t \rangle = \langle 0, 0, -18t^2 + 2 \rangle$. Thus,

$$\kappa(t) = \frac{|\vec{\mathbf{r}}\,' \times \vec{\mathbf{r}}\,''|}{|\vec{\mathbf{r}}\,'|^3} = \frac{|-18t^2 + 2|}{\left(\sqrt{(9t^2+1)^2 + (2t)^2} \right)^3}.$$

We have $\kappa(t) = 0$ if and only if $-18t^2 + 2 = 0$ or $t = \pm\frac{1}{3}$. Thus, there are two points along the curve at which $\kappa(t) = 0$.

Problem 4 (10 points) Assume the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are orthogonal to each other. Prove that $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$ is collinear to $\vec{\mathbf{b}}$. (Hint: rewrite $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}})$ using dot products.)

If $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are orthogonal, then $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$. By Formula 6 on p.841, we have

$$\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}} = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} - \vec{\mathbf{0}}.$$

Since $\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}$ is a scalar, this is a scalar multiple of $\vec{\mathbf{b}}$.

Problem 5 A "spaceship" leaves the origin at time $t = 0$. Its position at time t is given by the vector function $\vec{\mathbf{r}}(t) = \langle \frac{t^3}{3}, 8t - t^3, t^2 - \frac{t^3}{3} \rangle$.

(a) **(4 points)** At what time does the spaceship reach the plane $x + z = 4$?

We have $x + z = 4$ if and only if $\frac{t^3}{3} + t^2 - \frac{t^3}{3} = 4$ if and only if $t^2 = 4$. This happens when $t = \pm 2$. But since the shaceship leaves at $t = 0$, we pick the positive solution $t = 2$.

(b) **(8 points)** At what angle does the path of the spaceship intersect the plane from part (a)? (For your answer, it is enough to give the sine or the cosine of the angle. Make sure you indicate which one you are giving.)

The velocity vector of the spaceship is $\vec{\mathbf{r}}\,'(t) = \langle t^2, 8 - 3t^2, t(2 - t) \rangle$. Thus the velocity at $t = 2$, when the spaceship hits the plane, is $\vec{\mathbf{r}}\,'(2) = \langle 4, -4, 0 \rangle$. Let us call this vector $\vec{\mathbf{v}}$. A normal vector of the plane $x + z = 4$ is $\vec{\mathbf{n}} = \langle 1, 0, 1 \rangle$. If α is the angle between $\vec{\mathbf{v}}$ and $\vec{\mathbf{n}}$, and β is the angle between $\vec{\mathbf{v}}$ and the plane, then $\alpha + \beta = \pi/2$, and thus $\cos\alpha = \sin\beta$ (see illustration). We have

$$\sin\beta = \cos\alpha = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{n}}}{|\vec{\mathbf{v}}| \cdot |\vec{\mathbf{n}}|} = \frac{4}{\sqrt{32} \cdot \sqrt{2}} = \frac{1}{2}.$$

Thus $\beta = \pi/6$, or $\beta = 30°$.