MATH 582, INTRODUCTION TO SET THEORY, WINTER 1999
Answers to Problem Set 4

Problem 1 Recall that a binary relation $<$ on A is said to be a linear order (in the strict sense) if it is:

1. irreflexive: $\forall x \in A(\neg x < x)$,
2. transitive: $\forall x, y, z \in A(x < y \wedge y < z \Rightarrow x < z)$,
3. connected: $\forall x, y \in A(x < y \vee x = y \vee y < x)$.

A binary relation $\leqslant$ on A is said to be a linear order (in the non-strict sense) if it is:

1. reflexive: $\forall x \in A(x \leqslant x)$,
2. anti-symmetric: $\forall x, y \in A(x \leqslant y \wedge y \leqslant x \Rightarrow x = y)$,
3. transitive: $\forall x, y, z \in A(x \leqslant y \wedge y \leqslant z \Rightarrow x \leqslant z)$,
4. linear: $\forall x, y \in A(x \leqslant y \vee y \leqslant x)$.

Suppose $\langle A, < \rangle$ is a linear order in the strict sense, and define $x \leqslant y$ to mean $x < y \vee x = y$. We claim that $\langle A, \leqslant \rangle$ is a linear order in the non-strict sense. *Reflexivity:* for all x, one has $x = x$ and thus $x \leqslant x$. *Anti-symmetry:* Suppose $x \leqslant y$ and $y \leqslant x$. Assume, for the sake of contradiction, that $x \neq y$. Then $x < y$ and $y < x$, thus, $x < x$ by transitivity of $<$, contradicting irreflexivity of $<$. Thus $x = y$. *Transitivity:* Suppose $x \leqslant y$ and $y \leqslant z$. If $x = y$ then $x \leqslant z$ follows from $y \leqslant z$, and we are done. Similarly if $y = z$, then $x \leqslant z$ follows from $x \leqslant y$, and again we are done. The only case left to consider is when $x \neq y$ and $y \neq z$. But then $x < y < z$, and $x < z$, thus $x \leqslant z$, follows by transitivity of $<$. *Linearity:* Take $x, y \in A$. By connectedness of $<$, either $x < y$ or $x = y$ or $y < x$. In each of these cases, it follows that $x \leqslant y$ or $y \leqslant x$.

Conversely, suppose $\langle A, \leqslant \rangle$ is a linear order in the non-strict sense, and define $x < y$ to mean $x \leqslant y \wedge x \neq y$. We claim the $\langle A, < \rangle$ is a linear order in the strict sense. *Irreflexivity:* For any x, since $x \neq x$ does not hold, $x < x$ does not hold. *Transitivity:* Suppose $x < y$ and $y < z$. Then $x \leqslant y$ and $y \leqslant z$, and also $x \neq y$ and $y \neq z$. It follows that $x \leqslant z$ by transitivity of $\leqslant$. Assume that $x = z$, then $x = y$ follows by antisymmetry of $\leqslant$, a contradiction. Hence $x \neq z$, and thus $x < z$. *Connectedness:* Take $x, y \in A$. If $x = y$, we are done, so assume $x \neq y$. By linearity of $\leqslant$, we know that $x \leqslant y$ or $y \leqslant x$, and thus $x < y$ or $y < x$ by definition of $<$ and the fact that $x \neq y$.

Problem 2 First, assume that E is an equivalence relation on A, and consider any $x, y \in A$. We must show that $xEy \iff \forall z \in A(xEz \Rightarrow yEz)$. To prove the left-to-right half of this equivalence, assume xEy. Then yEx by symmetry, and therefore for all $z \in A$, xEz implies yEz by transitivity. To prove the right-to-left implication, assume that $\forall z \in A(xEz \Rightarrow yEz)$ holds. In particular, $xEx \Rightarrow yEx$. But xEx by reflexivity, and hence yEx, and by symmetry, xEy.

Conversely, assume that $xEy \iff \forall z \in A(xEz \Rightarrow yEz)$ holds for all $x, y \in A$. We want to show that E is an equivalence relation on A. *Reflexivity:* For any $x \in A$, the statement $\forall z \in A(xEz \Rightarrow xEz)$ is logically valid, and thus xEx by hypothesis. *Symmetry:* Suppose xEy. Then, by hypothesis, $\forall z \in A(xEz \Rightarrow yEz)$. In particular, $xEx \Rightarrow yEx$. But we have already proved that E is reflexive, so yEx holds as desired. *Transitivity:* Suppose xEy and yEz. The latter gives $\forall w \in A(yEw \Rightarrow zEw)$ by hypothesis, and in particular, $yEx \Rightarrow zEx$. We already know that E is symmetric, so we have yEx, and thus zEx. Another application of symmetry gives xEz, as desired.

Problem 3 Suppose R and Q are equivalence relations on sets A and B respectively, and $f : A \to B$ is a function. We want to prove that there exists a function $g : A/R \to B/Q$ satisfying $g([x]_R) = [f(x)]_Q$ for all $x \in A$, if and only if xRx' implies $f(x)\, Q\, f(x')$, for all $x, x' \in A$.

For the right-to-left implication, assume that g is such a function. If xRx', then $[x]_R = [x']_R$, thus $g([x]_R) = g([x']_R)$, thus $[f(x)]_Q = [f(x')]_Q$ by hypothesis, thus $f(x)\, Q\, f(x')$.

For the left-to-right implication, define $g = \{\langle [x]_R, [f(x)]_Q \rangle \mid x \in A\}$. Then g is certainly a relation of the appropriate type; we must check that it is a function. So consider any $\langle u, v \rangle, \langle u, w \rangle \in g$. By definition of g, there must be $x, x' \in A$ such that $u = [x]_R = [x']_R$, $v = [f(x)]_Q$, and $w = [f(x')]_Q$. It follows that xRx', hence $f(x)\, Q\, f(x')$ by hypothesis, hence $v = w$. This shows that g is a function. Do see that the domain of g is all of A/R, notice that for every $[x]_R \in A/R$, one has $\langle [x]_R, [f(x)]_Q \rangle \in g$, hence $[x]_R \in \operatorname{dom} g$. Thus, $g : A/R \to B/Q$. Moreover, it follows directly from the definition of g that it satisfies the desired property.

Problem 3.34 Assume that $\mathcal{A}$ is a non-empty set of transitive relations.

(a) The set $\bigcap \mathcal{A}$ is a transitive relation. Being a subset of some $A \in \mathcal{A}$, it is a relation. To show that it is transitive, take two pairs $\langle x, y\rangle, \langle y, z\rangle \in \bigcap \mathcal{A}$. Consider an arbitrary $A \in \mathcal{A}$. By definition of intersection, $\langle x, y\rangle, \langle y, z\rangle \in A$. Since A is a transitive relation, it follows that $\langle x, z\rangle \in A$. Since A was arbitrary, it follows that $\langle x, z\rangle \in \bigcap \mathcal{A}$.

(b) The set $\bigcup \mathcal{A}$ is not in general a transitive relation. The simplest counterexample is $\mathcal{A} = \{\{\langle 0, 1\rangle\}, \{\langle 1, 2\rangle\}\}$.

Problem 3.36 By definition, we have $Q \subseteq A \times A$, so Q is a relation on A. We want to show that Q is an equivalence relation on A. *Reflexivity:* For any $x \in A$, we have $\langle f(x), f(x)\rangle \in R$, by reflexivity of R. Thus, $\langle x, x\rangle \in Q$. *Symmetry:* Suppose $\langle x, y\rangle \in Q$. Then $\langle f(x), f(y)\rangle \in R$, thus $\langle f(y), f(x)\rangle \in R$ by symmetry of R. Hence $\langle y, x\rangle \in Q$. *Transitivity:* Suppose $\langle x, y\rangle, \langle y, z\rangle \in Q$. Then $\langle f(x), f(y)\rangle, \langle f(y), f(z)\rangle \in R$, hence $\langle f(x), f(z)\rangle \in R$ by transitivity of R. Thus $\langle x, z\rangle \in Q$.

Problem 3.41

(a) Q is an equivalence relation: *Reflexivity:* For any $\langle u, v\rangle \in \mathbb{R} \times \mathbb{R}$, $u + v = u + v$, and hence $\langle u, v\rangle Q \langle u, v\rangle$. *Symmetry:* Suppose $\langle u, v\rangle Q \langle x, y\rangle$. Then $u + y = x + v$, hence $x + v = u + y$, hence $\langle x, y\rangle Q \langle u, v\rangle$. *Transitivity:* Suppose $\langle u, v\rangle Q \langle x, y\rangle$ and $\langle x, y\rangle Q \langle w, z\rangle$. Then $u + y = x + v$ and $x + z = w + y$. Adding the two equations, we get $u + y + x + z = x + v + w + y$. Subtracting $x + y$, we obtain $u + z = w + v$, and thus $\langle u, v\rangle Q \langle w, z\rangle$.

(b) By Theorem 3Q (or Problem 3), what we must check is whether $\langle u, v\rangle Q \langle x, y\rangle$ implies $\langle u + 2v, v + 2u\rangle Q \langle x + 2y, y + 2x\rangle$. That is, we must check whether $u + y = x + v$ implies $(u + 2v) + (y + 2x) = (x + 2y) + (v + 2u)$. By adding $u + v + x + y$ to each side of the equation, and exchanging left and right sides, this does indeed follow. Therefore, the required G exists.

Problem 3.44 To show that f is one-to-one, take any $x \neq y$ in A; we have to show $f(x) \neq f(y)$. By connectedness of $<$, we have either $x < y$ or $y < x$. By hypothesis, this implies $f(x) < f(y)$ or $f(y) < f(x)$. In either case $f(x) \neq f(y)$ by irreflexivity. This shows that f is one-to-one.

Now assume that $f(x) < f(y)$; we want to show $x < y$. We cannot have $x = y$, because this would imply $f(x) = f(y)$, contradicting $f(x) < f(y)$ (by irreflexivity). Neither can we have $y < x$, because this would imply $f(y) < f(x)$ by hypothesis, and we would have $f(x) < f(x)$ by transitivity, again contradicting irreflexivity. Since $<$ is connected, the only possibility left is $x < y$.

Problem 3.45 Consider linearly ordered sets $\langle A, <_A\rangle$ and $\langle B, <_B\rangle$. The *lexicographic ordering* on $A \times B$ is the relation $<_L$ defined as follows:

$$\langle a_1, b_1\rangle <_L \langle a_2, b_2\rangle \qquad \text{iff} \qquad a_1 <_A a_2 \vee (a_1 = a_2 \wedge b_1 <_B b_2).$$

We show that $<_L$ is a linear ordering: *Irreflexivity:* Consider any $\langle a, b\rangle \in A \times B$. Since neither $a <_A a$ nor $b <_B b$ hold (by irreflexivity of $<_A$ and $<_B$), we do not have $\langle a, b\rangle <_L \langle a, b\rangle$. *Transitivity:* Suppose $\langle a_1, b_1\rangle <_L \langle a_2, b_2\rangle$ and $\langle a_2, b_2\rangle <_L \langle a_3, b_3\rangle$. We want to show $\langle a_1, b_1\rangle <_L \langle a_3, b_3\rangle$. Notice that $a_1 \leqslant a_2$ and $a_2 \leqslant a_3$. If we have $a_1 <_A a_2$ or $a_2 <_A a_3$, then $a_1 <_A a_3$, and we are done. Otherwise, $a_1 = a_2 = a_3$ and $b_1 <_B b_2 <_B b_3$. In this case, we have $b_1 <_B b_3$ by transitivity, and again, this implies $\langle a_1, b_1\rangle <_L \langle a_3, b_3\rangle$ as desired. *Connectedness:* Consider any $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle \in A \times B$. By connectedness of $<_A$, we have either $a_1 <_A a_2$ or $a_1 = a_2$ or $a_2 <_A a_1$. In the first case, we have $\langle a_1, b_1\rangle <_L \langle a_2, b_2\rangle$, and in the last case, we have $\langle a_2, b_2\rangle <_L \langle a_1, b_1\rangle$; if either of these happens, we are done. The remaining case is $a_1 = a_2$. By connectedness of $<_B$, we have either $b_1 <_B b_2$ or $b_1 = b_2$ or $b_2 <_B b_1$. In the first case, we have $\langle a_1, b_1\rangle <_L \langle a_2, b_2\rangle$, and in the last case, we have $\langle a_2, b_2\rangle <_L \langle a_1, b_1\rangle$. The only remaining case is when $b_1 = b_2$, but then we have $\langle a_1, b_1\rangle = \langle a_2, b_2\rangle$. Thus we are done.

Problem 6.22 We will show that the statement

For any set A there is a function $F : \bigcup A \to A$ such that $x \in F(x)$ for all $x \in \bigcup A$ (*)

is equivalent to the axiom of choice. We discussed four equivalent statements of the axiom of choice in class. Here, we will show (AC2) $\Rightarrow$ (*) $\Rightarrow$ (AC3).

To show the first implication, assume (AC2) and let A be any set. Consider the relation $R \subseteq (\bigcup A) \times A$ defined by $\langle x, a\rangle \in R \iff x \in a \in A$. By the definition of union, for each $x \in \bigcup A$, there exists $a \in A$ with $x \in a$; thus

$\bigcup A$ is the domain of R. By (AC2), there exists a function $F : \bigcup A \to A$ with $F \subseteq R$. This function F satisfies the property (*): namely, for all $x \in \bigcup A$, we have $\langle x, F(x) \rangle \in F \subseteq R$, thus $x \in F(x)$ by definition of R.

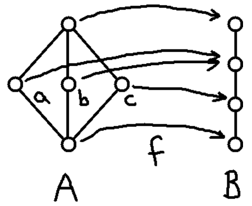
Now assume (*). We will show (AC3), i.e. the existence of a choice function $G : (\mathscr{P}A - \{\emptyset\}) \to A$, for any set A. So consider any set A. For any $z \in A$, define the set $I_z = \{a \subseteq A \mid z \in a\} \in \mathscr{P}\mathscr{P}A$. Let $A^* = \{I_z \mid z \in A\} \subseteq \mathscr{P}\mathscr{P}A$. Let $\phi : A \to A^*$ be the function defined by $\phi(z) = I_z$. We claim:

1. ϕ is one-to-one and onto. Thus $\phi^{-1} : A^* \to A$ is a well-defined function. Proof: To show that ϕ is one-to-one, assume $\phi(y) = \phi(z)$ for some $y, z \in A$. Then $I_y = I_z$. Since $\{y\}$ is a member of I_y, it must also be a member of I_z, which means, by definition of I_z, that $z \in \{y\}$, hence $y = z$. This proves that ϕ is one-to-one. Clearly, ϕ is onto A^* by definition of A^*.

2. $\bigcup A^* = \mathscr{P}A - \{\emptyset\}$. Proof: $a \in \bigcup A^*$ iff $\exists z \in A(a \in I_z)$ iff $\exists z \in a \subseteq A$ iff $a \subseteq A$ and $a \neq \emptyset$ iff $a \in \mathscr{P}A - \{\emptyset\}$.

By the hypothesis (*), when applied to the set A^*, it follows that there exists a function $F : \bigcup A^* \to A^*$ such that $a \in F(a)$ for all $a \in \bigcup A^*$. We define new define a function $G : (\mathscr{P}A - \{\emptyset\}) \to A$ as follows: Let $G(a) = \phi^{-1}(F(a))$. By claims 1 and 2 above, this is a well-defined function. Now consider any $a \in \mathscr{P}A - \{\emptyset\}$. Let $z = G(a)$. Then $I_z = \phi(z) = \phi(G(a)) = \phi(\phi^{-1}(F(a))) = F(a)$ by definitions of ϕ and G. Also $a \in F(a)$ by construction of F, hence $a \in I_z$. The latter implies $z \in a$ by definition of I_z. Since z was $G(a)$, we have $G(a) \in a$. Since a was arbitrary, this holds for all $a \in \mathscr{P}A - \{\emptyset\}$, which proves that G is the desired choice function.

Problem 7.1 Unlike in Problem 3.44, both claims are wrong for partial orders. Consider the following function between partially ordered sets:



This function satisfies $x <_A y \Rightarrow f(x) <_B f(y)$, but it is neither one-to-one, nor does it satisfy $f(x) <_B f(y) \Rightarrow x <_A y$. Notice that the elements b and c satisfy $f(c) <_B f(b)$, but not $c <_A b$.