MATH 582, INTRODUCTION TO SET THEORY, WINTER 1999

Answers to Problem Set 6 (Revised)

Problem 4.19 We want to prove the claim: for all natural numbers d and m, if $d \neq 0$, then there exists natural numbers q, r such that $m = d \cdot q + r$ and $r < d$. Fix some $d \neq 0$; we will prove the claim by induction on m. For the *base case*, notice that if $m = 0$, then we can take $q = r = 0$. Indeed, we have $0 = d \cdot 0 + 0$ by (M1) and (A1), and also $0 < d$, because $d \neq 0$. For the *induction step*, assume the claim holds for m. So there exist some $q, r \in \omega$ with $m = d \cdot q + r$ and $r < d$. Note that this implies $r^+ \leqslant d$. We distinguish two cases: Case 1: $r^+ < d$. In this case, we have $m^+ = d \cdot q + r^+$ by (A2), and we are done. Case 2: $r^+ = d$. In this case, we have $m^+ = d \cdot q + d = d \cdot q^+$ by (M2), so $m^+ = d \cdot q^+ + 0$, and since $0 < d$, we are also done.

Problem 4.20 Recall from p.83 that for natural numbers n and k, one has $k \in n \iff k \in n^+$. By taking the negations of these statements and using trichotomy, one also gets that $n \in k \iff n^+ \in k$. We will use this fact in the following proof.

Suppose A is a nonempty subset of ω and $\bigcup A = A$. To prove $A = \omega$, we show that A is inductive. *Base case:* Since A is nonempty, there is some natural number $k \in A$. Either $k = 0$, in which case we have $0 \in A$ as desired. Otherwise, $0 \in k \in A$, hence $0 \in \bigcup A = A$. *Induction step:* Suppose $n \in A$. Then, since $A = \bigcup A$, we have $n \in \bigcup A$, which implies $n \in k$ for some $k \in A$. Since A is a set of natural numbers, k is a natural number. By the above observation, we have $n^+ \in k$. There are two cases: Case 1: $n^+ = k$, thus $n^+ \in A$ and we are done. Case 2: $n^+ \in k \in A$, hence $n^+ \in \bigcup A = A$, and again we are done. Thus A is inductive, hence $A = \omega$.

Problem 4.26 We proceed by induction on n. Let

$$T = \{n \in \omega \mid \text{for all } f : n^+ \to \omega, \operatorname{ran} f \text{ has a largest element}\}.$$

We will show that T is inductive. First consider any $f : 0^+ \to \omega$. Since $0^+ = \{0\}$ is a singleton, $\operatorname{ran} f = \{f(0)\}$ has exactly one element, which is automatically largest. This takes care of the base case. Now suppose that $n \in T$. To show that $n^+ \in T$, consider any $f : n^{++} \to \omega$. We have to show that $\operatorname{ran} f$ has a largest element. Let $g = f|n^+$ be the restriction of f to n^+; thus $g : n^+ \to \omega$. By induction hypothesis, we know that $\operatorname{ran} g$ has a largest element, say, k. Also, $\operatorname{ran} f = \operatorname{ran} g \cup \{f(n^+)\}$. We distinguish two cases: Case 1: $k < f(n^+)$. In this case, $f(n^+)$ is the largest element of $\operatorname{ran} f$. Case 2: $f(n^+) \leqslant k$. In this case, k is the largest element of $\operatorname{ran} f$. In either case, $\operatorname{ran} f$ has a largest element, and since f was arbitrary, it follows that $n^+ \in T$. Thus, T is inductive, which proves the claim.

Lemma. *If $n, k \in \omega$ and $n \in k$, then there exists $x \in \omega$ with $k = x + n$.*

Proof. By induction on k. If $k = 0$ then $n \in k$ implies $n = 0$, and we can take $x = 0$. For the induction step, suppose the claim holds for k, and suppose $n \in k^+$. Then either $n = k^+$, in which case one can take $x = 0$. Or otherwise $n \in k^+$, thus $n \in k$, and we can find x with $k = x + n$ by induction hypothesis. Then $k^+ = x^+ + n$, and we are done. □

Problem 4.37

(a) For fixed $m, n \in \omega$, define the set $M = \{k \in m + n \mid k \notin n\}$. We claim that M has m elements. Let $\phi : m \to M$ be the map that is defined by $\phi(x) = x + n$. We must show that ϕ is a well-defined map, and that it is a bijection. To see that ϕ is well-defined, we must check that $x + n \in M$ for all $x \in m$. Note that if $x \in m$, then $x + n \in m + n$ by Theorem 4N. Also, $x \notin 0$ and thus $x + n \notin 0 + n = n$, again by Theorem 4N. Thus, $x + n \in M$, and ϕ is well-defined. Also, ϕ is one-to-one by Corollary 4P. To see that ϕ is onto M, take any $k \in M$. Then $k \notin n$, thus $n \in k$ by trichotomy. By the Lemma, there exists $x \in \omega$ with $k = x + n$. Since $k = x + n \in m + n$, it follows that $x \in m$ by Theorem 4N, thus $k = \phi(x)$. This shows that ϕ is onto.

Notice that M and n are disjoint, by definition of M. We claim that $m + n = M \cup n$. For the right-to-left inclusion, notice that $M \subseteq m + n$ by definition. Also, $0 \in m$, thus $n = n + 0 \in n + m$ by Theorem 4N and Corollary 4P, thus $n \subseteq n + m$ by Corollary 4M. So we have $M \cup n \subseteq m + n$. For the left-to-right inclusion, take any $k \in m + n$. In case $k \in n$, we are done, otherwise $k \notin n$, which implies $k \in M$. This shows $m + n \subseteq M \cup n$.

Now we show the claim of part (a). Assume A, B are disjoint of m, respectively n, elements. Let $f : m \to A$ and $g : n \to B$ be bijections, and let $\phi : m \to M$ be the bijection from above. Let $h : M \to A$ be the bijection given by $h = f \circ \phi^{-1}$. Using the fact that M and n are disjoint and that A and B are disjoint, it follows easily that $h \cup g : M \cup n \to A \cup B$ is a bijection. From $m + n = M \cup n$, it follows that $A \cup B$ has $m + n$ elements.

(b) We first claim that $m \times n$ has $m \cdot n$ elements. Define $\psi : m \times n \to m \cdot n$ by $\psi(x, y) = x \cdot n + y$. We claim that ψ is well-defined, and that it is a bijection. For well-definedness, we must check that $\psi(x, y) \in m \cdot n$ whenever $x \in m$ and $y \in n$. But $x \in m$ implies $x^+ \in\!\!\!= m$, and thus $x^+ \cdot n \in\!\!\!= m \cdot n$. The last step follows by Theorem 4N, if $n \neq 0$, and by (M1) if $n = 0$. Now $y \in n$ implies $x \cdot n + y \in x \cdot n + n = x^+ \cdot n \in\!\!\!= m \cdot n$. Thus $\psi(x, y) \in m \cdot n$, and ψ is well-defined.

To show that ψ is one-to-one, assume that $\psi(x, y) = \psi(x', y')$ for some $x, x' \in m$ and $y, y' \in n$. Then $x \cdot n + y = x' \cdot n + y'$. We must show $x = x'$ and $y = y'$. First, assume (for the sake of deriving a contradiction) that $x \neq x'$. Then either $x \in x'$ or $x' \in x$ by trichotomy; we may assume without loss of generality that $x \in x'$. It follows that $x^+ \in\!\!\!= x'$, thus $x^+ \cdot n \in\!\!\!= x' \cdot n$. This implies $x \cdot n + y \in x \cdot n + n = x^+ \cdot n \in\!\!\!= x' \cdot n \in\!\!\!= x' \cdot n + y'$, contradicting $x \cdot n + y = x' \cdot n + y'$. Thus, it follows that $x = x'$. Now from $x \cdot n + y = x \cdot n + y'$ we can get $y = y'$ by cancellation (Cor. 4P).

Next, we show that ψ is onto $m \cdot n$. If $n = 0$, then this is trivial, since $m \cdot n = 0$ in this case. Thus, assume $n \neq 0$ and take any $k \in m \cdot n$. By Problem 4.19, there exist numbers x and y such that $k = x \cdot n + y$ and $y \in n$. We have $x \cdot n \in\!\!\!= k \in m \cdot n$, and thus $x \in m$ by Theorem 4N. It follows that $k = \psi(x, y)$. Thus ψ is a bijection.

Finally, we show the claim of part (b). Assume A has m elements and B has n elements. Let $f : m \to A$ and $g : n \to B$ be bijections. Define $h : m \times n \to A \times B$ by $h(\langle x, y\rangle) = \langle f(x), g(y)\rangle$. One checks easily that h is a bijection. Then $h \circ \psi^{-1}$ is a bijection $m \cdot n \to A \times B$, which proves that $A \times B$ has $m \cdot n$ elements.

Here is an alternative, easier proof of Problem 4.37 which uses induction.

(a) We show this claim by induction on n. If $n = 0$, then $B = \emptyset$, and hence $A \cup B = A$ has $m = m + n$ elements. For the induction step, assume the claim holds for n. Suppose A has m elements, B has n^+ elements, and A and B are disjoint. Then there exists some one-to-one and onto function $f : n^+ \to B$. Let $B' = f[\![n]\!]$; then clearly $B = B' \cup f(n)$ and this union is disjoint. Moreover, the function $f|_n : n \to B$ is one-to-one and onto B', so that B' has n elements. Since B' is still disjoint from A, the set $A \cup B'$ has $m + n$ elements by induction hypothesis. Thus, there is some one-to-one and onto function $g : m + n \to A \cup B'$. Let $h = g \cup \langle m + n, f(n)\rangle$, then h is a one-to-one function of $m + n^+$ onto $A \cup B$, as desired.

(b) Again, we show the claim by induction on n. If $n = 0$, then $B = \emptyset$, and hence $A \times B = \emptyset$, which has $0 = m \cdot n$ elements. For the induction step, assume the claim holds for n. Suppose A has m elements and B has n^+ elements. Then there exists some one-to-one and onto function $f : n^+ \to B$. As before, let $B' = f[\![n]\!]$; then again $B = B' \cup f(n)$, this union is disjoint, and B' has n elements. By induction hypothesis, $A \times B'$ has $m \cdot n$ elements. Since A has m elements, and the set A is in one-to-one correspondence with the set $A \times \{f(n)\}$, the latter set also has m elements. One also has $A \times B = A \times B' \cup A \times \{f(n)\}$ (by Problem 3.2(a)), moreover, the latter union is disjoint, and so by (a), $A \times B$ has $m \cdot n + m = m \cdot n^+$ elements, as desired.

Problem 5.1 through 5.3 We must check whether each of the following functions from $\omega \times \omega$ to $\omega \times \omega$ is compatible with the relation $\sim$, which was defined by $\langle m, n\rangle \sim \langle m', n'\rangle$ iff $m + n' = m' + n$.

$$\begin{aligned} f(\langle m, n\rangle) &= \langle m + n, n\rangle, \\ g(\langle m, n\rangle) &= \langle m, m\rangle, \\ h(\langle m, n\rangle) &= \langle n, m\rangle. \end{aligned}$$

The function f is not compatible: for instance $\langle 0, 0\rangle \sim \langle 1, 1\rangle$, but $\langle 0 + 0, 0\rangle \not\sim \langle 1 + 1, 1\rangle$. The function g is trivially compatible, because for all m, m', one has $\langle m, m\rangle \sim \langle m', m'\rangle$. The function h is also compatible, because $\langle m, n\rangle \sim \langle m', n'\rangle$ implies $m + n' = m' + n$ implies $n + m' = n' + m$ implies $\langle n, m\rangle \sim \langle n', m'\rangle$.

Problem 5.7 Let $a = [\langle x, y\rangle]$ and $b = [\langle w, z\rangle]$. Then

$$\begin{array}{rcl}
a \cdot_{\mathbb{Z}} (-b) & = & [\langle x, y\rangle] \cdot_{\mathbb{Z}} (-[\langle w, z\rangle]) \\
& = & [\langle x, y\rangle] \cdot_{\mathbb{Z}} [\langle z, w\rangle] \\
& = & [\langle xz + yw, xw + yz\rangle], \\
(-a) \cdot_{\mathbb{Z}} b & = & (-[\langle x, y\rangle]) \cdot_{\mathbb{Z}} [\langle w, z\rangle] \\
& = & [\langle y, x\rangle] \cdot_{\mathbb{Z}} [\langle w, z\rangle] \\
& = & [\langle yw + xz, yz + xw\rangle] \\
& = & [\langle xz + yw, xw + yz\rangle], \\
-(a \cdot_{\mathbb{Z}} b) & = & -([\langle x, y\rangle] \cdot_{\mathbb{Z}} [\langle w, z\rangle]) \\
& = & -([\langle xw + yz, xz + yw\rangle]) \\
& = & [\langle xz + yw, xw + yz\rangle].
\end{array}$$