MATH 316, DIFFERENTIAL EQUATIONS, WINTER 2000

Answers to the First Midterm

Problem 1 (12 points) Suppose the temperature of a cup of coffee obeys Newton's law of cooling. A cup of coffee is freshly poured and placed outdoors on a cold day. Suppose the coffee has a temperature of 200°F when it is first poured, and after ten minutes is has cooled down to 146°F. If the outside temperature is 38°F, then at what time does the coffee reach 86°F? (You may use the facts that $\ln\frac{2}{3} \approx -0.4$ and that $\ln\frac{8}{27} \approx -1.2$).

Answer: Let y be the temperature, in $^\circ$F, as a function of time t, in minutes. Newton's law of cooling states that y' is proportional to the temperature difference, so $y' = k(y-38)$. Separating, we get $dy/(y-38) = k\,dt$. Integrating, we obtain $\ln|y-38| = kt + C$, or $y - 38 = Ae^{kt}$, where A is any constant. We plug in the first initial condition $t = 0$ and $y = 200$, to obtain $A = 162$. Next, we plug in the second "initial" condition $t = 10$ and $y = 146$ to obtain $146 - 38 = 162e^{10k}$, or $k = \frac{1}{10}\ln\frac{108}{162} = \frac{1}{10}\ln\frac{2}{3}$. Finally, we set $y = 86$ and solve for t. We get $86 - 38 = 162e^{kt}$, or $t = \frac{1}{k}\ln\frac{48}{162} = 10\frac{\ln\frac{8}{27}}{\ln\frac{2}{3}} = 10 \cdot 3 = 30$. Thus the coffee reaches 86°F thirty minutes after it is poured.

Problem 2 (12 points) For each of the following differential equations, state its type (for instance linear, non-linear, separable, exact, homogeneous) and find the general solution to each equation. (You do not need to solve your answer for y).

(a) $\frac{dy}{dx} = \frac{x^2}{y^2} + \frac{y}{x}$.

Answer: This equation is homogeneous. If we do the substitution $y = vx$ (thus $y' = v'x + v$), we get $v'x + v = 1/v^2 + v$, or $v^2 dv = dx/x$. Integrating, we obtain $v^3/3 = \ln|x| + C$. Resubstituting $v = y/x$, we get $\frac{y^3}{3x^3} = \ln|x| + C$.

(b) $y' + y = t$.

Answer: This is a linear equation of the form $y' + p(t)y = g(t)$, where $p(t) = 1$, $g(t) = t$. We find an integrating factor $\mu(t) = \exp\int p(t)dt = e^t$. We calculate $\int \mu(t)g(t)dt = \int te^t dt = te^t - e^t + C$ (using integration by parts). Thus the general solution is

$$y = \frac{\int \mu(t)g(t)dt + C}{\mu(t)} = \frac{te^t - e^t + C}{e^t} = t - 1 + Ce^{-t}.$$

(c) $(x^2 + 2x^3y)\frac{dy}{dx} + 2xy + 3x^2y^2 = 0$.

Answer: This is an exact equation, because $\frac{d}{dx}(x^2 + 2x^3y) = 2x + 6x^2y = \frac{d}{dy}(2xy + 3x^2y^2)$. We find the potential $\phi(x,y) = x^2y + x^3y^2$. The solution is $\phi(x,y) = C$, or $x^2y + x^3y^2 = C$.

Problem 3 (12 points) Suppose that the size y of a certain fish population in a pond is governed by the differential equation

$$y' = y(1 - \frac{y}{400}) - R.$$

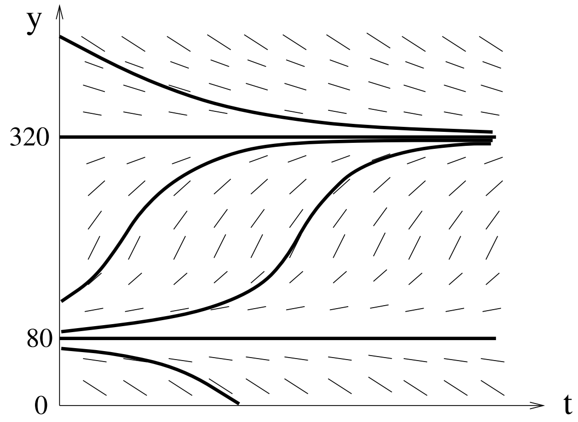
Here R is the rate at which fish are being removed from the pond (and turned into cat food by a nearby cannery).

(a) Suppose the cannery harvests fish at a rate of $R = 64$. Sketch a slope field for the differential equation. Make sure you indicate the scale on the y-axis. In your slope field, draw all equilibrium solutions, and at least two increasing and two decreasing solution curves.

Answer: We first determine the equilibrium solutions by setting $y' = 0$:

$$\begin{aligned} y(1 - y/400) - 64 = 0 \quad &\Leftrightarrow \quad -y^2/400 + y - 64 = 0 \\ &\Leftrightarrow \quad y = \tfrac{-1 \pm \sqrt{1 - 64/100}}{-2/400} \\ &\Leftrightarrow \quad y = 80 \text{ or } y = 320. \end{aligned}$$

Moreover, y' is positive for $80 < y < 320$, and negative for $y < 80$ and $320 < y$. Thus we get the following sketch:



(b) With $R = 64$, what will be the size of the fish population in the long run (as $t \to \infty$) if $y(0) = 100$? What if $y(0) = 50$?

Answer: From the slope field, you can see that y will tend towards the stable equilibrium (y=320) if the initial condition is $y(0) = 100$. If the initial condition is $y(0) = 50$, the fish population will go to zero.

(c) What is the maximal rate R at which the cannery can harvest fish without risking the extinction of the fish population?

Answer: If the harvest rate is R, then the equilibrium solutions are determined by $y = \frac{-1 \pm \sqrt{1 - R/100}}{-2/400}$ as in part (a). By looking at the term under the square root, we find that when $R > 100$, then there is no equilibrium solution and y' is always negative, whereas when $R \leqslant 100$, there are some equilibrium solutions. Thus $R = 100$ is the maximum harvest rate at which the fish population has a chance to survive.