MATH 316, DIFFERENTIAL EQUATIONS, WINTER 2000

Answers to the Final Exam

Problem 1 (9 points) A tank initially contains 200 liters of pure water. A mixture containing a concentration of 6 grams/liter of salt enters the tank at a rate of 2 liters/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression for the amount of salt in the tank at time t. How much salt is in the tank after 120 minutes? What is the limiting amount of salt in the tank as $t \to \infty$? (You may use the fact that $e^{-1.2} \approx 0.3$).

Answer: If S is the amount of salt in the tank, then $S' = in - out = 6 \cdot 2 - \frac{S}{200} \cdot 2 = \frac{1200-S}{100}$. Solving this, we get $S(t) = 1200 - Ae^{-\frac{t}{100}}$. Plugging in the initial condition $S(0) = 0$, we get $A = 1200$, thus $S(t) = 1200 - 1200e^{-\frac{t}{100}}$. At $t = 120$, we have $S(120) = 1200 - 1200e^{-\frac{120}{100}} \approx 1200 - 1200 \cdot 0.3 = 1200 - 360 = 840$. As $t \to \infty$, we have $e^{-\frac{t}{100}} \to 0$ and thus $S(t) \to 1200$.

Problem 2 (9 points) For each of the following differential equations, state its order and type (for instance, linear, non-linear, separable, exact, homogeneous, etc). Find the general solution to each equation.

(a) $\dfrac{dy}{dt} = \dfrac{y}{t} + \dfrac{y^2}{t^2}$.

Answer: This equation is first-order and homogeneous. We do the standard substitution $y = vt$, $y' = v + v't$ to get $v + v't = v + v^2$, thus $v' = v^2/t$. This is now separable: $dv/v^2 = dt/t$, thus $-1/v = \ln t + C$, or $v = -1/(\ln t + C)$. Substituting back for $y = vt$, we get $y = -t/(\ln t + C)$.

(b) $\dfrac{1}{t}\dfrac{dx}{dt} = x(t^2 - 1)$.

Answer: This equation is first-order and separable. We have $dx/x = (t^3 - t)dt$, thus $\ln|x| = t^4/4 - t^2/2 + C$, thus $x = \pm e^{t^4/4 - t^2/2 + C}$ or $x = Ae^{t^4/4 - t^2/2}$.

(c) $(2x^3y + x^4)\dfrac{dy}{dx} + 4x^3y = -3x^2y^2$.

Answer: This equation is first-order. It is of the form $N(x,y)\frac{dy}{dx} + M(x,y) = 0$, where $N(x,y) = 2x^3y + x^4$ and $M(x,y) = 4x^3y + 3x^2y^2$. We check that the equation is exact: $\partial N/\partial x = 6x^2y + 4x^3$ and $\partial M/\partial y = 4x^3 + 6x^2y$. We find the common antiderivative $\psi(x,y) = x^3y^2 + yx^4$. Thus, the general solution is $\psi(x,y) = C$, or $x^3y^2 + yx^4 = C$.

(d) $y'' = y' + 2y$.

Answer: This equation is second-order and linear. Assuming solutions of the form $y = e^{rt}$, we get the condition $r^2 = r + 2$, which is satisfied for $r = 2$ and $r = -1$. Thus $y = e^{2t}$ and $y = e^{-t}$ are two linearly independent solutions. The general solution is $y = c_1e^{2t} + c_2e^{-t}$.

Problem 3 (9 points)

(a) Find three linearly independent solutions of the system

$$\frac{d}{dt}\vec{x} = \begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \vec{x}.$$

Answer: First, we find the eigenvalues of the matrix. The eigenpolynomial is $(-1-\lambda)(-\lambda)(2-\lambda)-(-1-\lambda)(-1) = (-1-\lambda)(\lambda-1)^2$. Thus, we have an eigenvalue $\lambda_1 = -1$ of multiplicity one and an eigenvalue $\lambda_2 = 1$ of multiplicity two. The corresponding eigenvectors are: for $\lambda_1 = -1$, $v_1 = (1,0,0)^\top$, and for $\lambda_2 = 1$, $v_2 = (1,-1,1)^\top$. Also, there is a generalized eigenvector for $\lambda_2 = 1$, which is $v_3 = (0,0,1)^\top$. Thus, three linearly independent solutions are:

$$\vec{x}(t) = \begin{pmatrix}1\\0\\0\end{pmatrix} e^{-t} \quad \text{and} \quad \vec{x}(t) = \begin{pmatrix}1\\-1\\1\end{pmatrix} e^{t} \quad \text{and} \quad \vec{x}(t) = \begin{pmatrix}1\\-1\\1\end{pmatrix} te^{t} + \begin{pmatrix}0\\0\\1\end{pmatrix} e^{t}.$$

(b) Determine the solution which satisfies the initial condition

$$\vec{x}(0) = \begin{pmatrix}1\\2\\1\end{pmatrix}.$$

Answer: The general solution is

$$\vec{x}(t) = c_1 \begin{pmatrix}1\\0\\0\end{pmatrix} e^{-t} + c_2 \begin{pmatrix}1\\-1\\1\end{pmatrix} e^{t} + c_3 \left[\begin{pmatrix}1\\-1\\1\end{pmatrix} te^{t} + \begin{pmatrix}0\\0\\1\end{pmatrix} e^{t}\right].$$

Plugging in the initial condition, we get

$$\begin{pmatrix}1\\2\\1\end{pmatrix} = c_1 \begin{pmatrix}1\\0\\0\end{pmatrix} + c_2 \begin{pmatrix}1\\-1\\1\end{pmatrix} + c_3 \begin{pmatrix}0\\0\\1\end{pmatrix},$$

which yields $c_1 = 3$, $c_2 = -2$, $c_3 = 3$.

Problem 4 (9 points) Find the general solution of the second-order linear differential equation

$$t^2y'' + 6ty' + 4y = 4 + 10t.$$

Hint: For the homogeneous part, try solutions of the form t^r, and for the non-homogeneous part, try solutions of the form $A + Bt$.

Answer: We first solve the homogeneous equation $t^2y''+6ty'+4y = 0$. Trying $y = t^r$, we get the condition $t^2r(r-1)t^{r-2} + 6trt^{r-1} + 4t^r = 0$. Dividing by t^r we get $r(r-1)+6r+4 = 0$, or $r^2+5r+4 = 0$, which is satisfied when $r = -1$ or $r = -4$. Thus the general solution of the homogeneous system is $y = c_1t^{-1} + c_2t^{-4}$. We now look for a particular solution of the non-homogeneous system of the form $y = A + Bt$. Plugging this into the non-homogeneous equation, we get $6tB + 4(A + Bt) = 4 + 10t$, or $10Bt + 4A = 4 + 10t$. This is satisfied when $A = 1$ and $B = 1$. Thus, $y = 1 + t$ is a particular solution. The general solution of the non-homogeneous system is $y = c_1t^{-1} + c_2t^{-4} + 1 + t$.

Problem 5 (9 points) Calculate e^A, where A is the following matrix:

$$A = \begin{pmatrix}2 & 1\\0 & 1\end{pmatrix}$$

Answer: We use diagonalization. The eigenvalues are 2 and 1, and the corresponding eigenvectors are $(1,0)^\top$ and $(-1,1)^\top$, respectively. Then $D = T^{-1}AT$ is a diagonal matrix, where

$$T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus,

$$e^A = Te^DT^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^2 & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^2 & e^2 - e \\ 0 & e \end{pmatrix}.$$

Problem 6 (9 points) Find and classify all equilibrium solutions of the following system, and sketch the phase portrait. Include the nullclines as well.

$$\frac{dx}{dt} = x(y - y^2),$$
$$\frac{dy}{dt} = (y+1)(x-1).$$

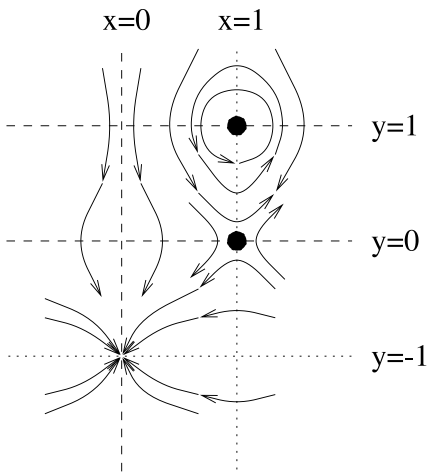
Answer: We see that $dx/dt = 0$ if $x = 0$ or $y = 0$ or $y = 1$. These nullclines are shown as dashed lines in the sketch below. Also, $dy/dt = 0$ if $y = -1$ or $x = 1$. These nullclines are shown as dotted lines. The equilibrium points are where the dashed and the dotted lines intersect; this happens at the three points $(1,1)$, $(1,0)$, and $(0,-1)$. To analyze the three equilibrium points, we calculate the Jacobian

$$J(x,y) = \begin{pmatrix} y - y^2 & x(1-2y) \\ y+1 & x-1 \end{pmatrix}.$$

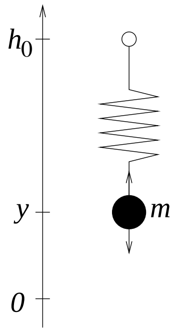
At the three points, the Jacobian evaluates to:

$$J(1,1) = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}, \qquad J(1,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad J(0,-1) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

At $(1,1)$, the Jacobian has complex eigenvalues (with zero real part), giving rise to a counterclockwise rotation. At $(1,0)$, the eigenvalues are 1 and -1 with respective eigenvectors $(1,1)^\top$ and $(1,-1)^\top$, giving rise to a saddle point. At $(0,-1)$, the eigenvalues are -2 and -1 with respective eigenvectors $(1,0)^\top$ and $(0,1)^\top$, giving rise to a stable equilibrium point.



Problem 7 (9 points) A mass m is suspended from a positive height h_0 by a spring, as shown in the illustration. The mass can only move up and down (not to the right and left). The spring constant is k, and the spring has relaxed length l_0. Let y be the vertical height of the mass.



(a) How many degrees of freedom does this mechanical system have?

Answer: 1, the height of the mass.

(b) Find formulas for the potential energy U and the kinetic energy T of the system in terms of y and $\dot{y}$.

Answer:

$$U = mgy + \frac{k}{2}((h_0 - y) - 1)^2$$

$$T = \frac{m}{2}\dot{y}^2.$$

(c) Use the Lagrangian formalism to find the equations of motion of this system.

Answer:

$$L = T - U = \frac{m}{2}\dot{y}^2 - mgy - \frac{k}{2}((h_0 - y) - l_0)^2.$$

We calculate

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} &= \frac{d}{dt}m\dot{y} = m\ddot{y}, \\ \frac{\partial L}{\partial y} &= -mg + k(h_0 - y - l_0). \end{aligned}$$

Thus the equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \qquad \text{or} \qquad m\ddot{y} = -mg + k(h_0 - y - l_0).$$