REPRESENTATION 2-CATEGORIES OF 2-GROUPS

Josep Elgueta

International Category Theory Conference 2006

Contents of the talk

1. Review on 2-groups.

2. Bicategory of representations of a 2-group over an arbitrary bicategory $\mathfrak{C}$.

3. Kapranov and Voevodsky 2-vector spaces

4. Representation theory on $\mathsf{2Vect}_K$

5. Reasons to study representations of 2-groups

1. Review on 2-groups

Definition. A *2-group* (or *categorical group*) is a one object bigroupoid.

Notation:

- 2-group as a monoidal category = $\mathbb{G}$
- 2-group as a one object bigroupoid = $\mathbb{G}[1]$

Example: *Split 2-groups* $G[0] * A[1]$, with G any group and A any G-module

$$(g, a') \circ (g, a) = (g, a' + a)$$
$$g_1 \otimes g_2 = g_1 g_2$$
$$(g_1, a_1) \otimes (g_2, a_2) = (g_1 g_2, a_1 + (g_1 \cdot a_2))$$

Classification theorem. (Sinh, 1975) *There is a bijection*

$$\left\{\begin{array}{c}\text{Equivalence}\\ \text{classes of}\\ \text{2-groups}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{equivalence classes}\\ \text{of triples } (G, A, [\alpha])\\ \text{with:}\\ \quad\text{- } G \text{ a group}\\ \quad\text{- } A \text{ a } G\text{-module}\\ \quad\text{- } [\alpha] \in H^3(G, A)\end{array}\right\}$$

$$\mathbb{G} \quad \mapsto \quad (\pi_0(\mathbb{G}), \pi_1(\mathbb{G}), [\alpha])$$

- $\pi_0(\mathbb{G})$ = iso- classes of objects in $\mathbb{G}$
  (a group with $[A][A'] = [A \otimes A']$)

- $\pi_1(\mathbb{G}) = \mathsf{Aut}_{\mathbb{G}}(I)$
  (it has a $\pi_0(\mathbb{G})$-module structure)

- $[\alpha] \in H^3(\pi_0(\mathbb{G}), \pi_1(\mathbb{G}))$ is determined by the components of the associator
  $$\alpha([A_1], [A_2], [A_3]) \sim f(\mathsf{a}_{A_1, A_2, A_3})$$

Theorem. (Baez-Lauda, 2004) *For any 2-groups $\mathbb{G}$ and $\mathbb{G}'$, let α, α' be classifying 3-cocycles of $\mathbb{G}, \mathbb{G}'$. Then, there is a bijection*

$$\left\{ \begin{array}{c} \text{Isomorphism} \\ \text{classes of} \\ \text{2-group} \\ \text{morphisms} \\ \mathbb{G} \xrightarrow{F} \mathbb{G}' \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Triples } (\rho, \beta, [c]) \\ \text{with:} \\ \text{- } \rho : \pi_0(\mathbb{G}) \to \pi_0(\mathbb{G}') \text{ a} \\ \qquad \text{group morphism} \\ \text{- } \beta : \pi_1(\mathbb{G}) \to \pi_1(\mathbb{G}')_\rho \\ \qquad \text{a } \pi_0(\mathbb{G})\text{-module} \\ \qquad \text{morphism such that} \\ \qquad [\beta\ \alpha] = [\alpha'\ \rho^3] \\ \text{- } [c] \in \widetilde{H}^2(\pi_0(\mathbb{G}), \pi_1(\mathbb{G}')_\rho) \\ \qquad \text{such that} \\ \qquad \partial c = \beta\ \alpha - \alpha'\ \rho^3 \end{array} \right\}$$

where $\widetilde{H}^n := C^n / B^n$.

2. Bicategory of representations of a 2-group $\mathbb{G}$ over a bicategory $\mathfrak{C}$

Recall: for any group G and any category $\mathcal{C}$

$$\mathbf{Rep}_{\mathcal{C}}(G) \stackrel{\text{def}}{=} [G[1], \mathcal{C}]$$

with $G[1]$ = one object groupoid defined by G.

Definition. Let $\mathbb{G}$ be any 2-group and $\mathfrak{C}$ any bicategory. The bicategory of representations of $\mathbb{G}$ on $\mathfrak{C}$ is the bicategory

$$\boxed{\mathsf{Rep}_{\mathfrak{C}}(\mathbb{G}) \stackrel{\text{def}}{=} [\mathbb{G}[1], \mathfrak{C}]}$$

with $\mathbb{G}[1]$ = one object bigroupoid defined by $\mathbb{G}$

Example. For any category $\mathcal{C}$, let $\mathcal{C}[0]$ be the corresponding (locally) discrete 2-category. Then

$$\boxed{\mathsf{Rep}_{\mathcal{C}[0]}(\mathbb{G}) \;=\; \mathbf{Rep}_{\mathcal{C}}(\pi_0(\mathbb{G}))[0]}$$

(slightly better than for groups, because for any group G and set X it is $\mathbf{Rep}_{X[0]}(G) = X[0]$...)

Key point:

What 2-category $\mathfrak{C}$ should we choose to get a good theory of representations for 2-groups?

3. Kapranov and Voevodsky 2-vector spaces

Definition (KV, 94). A *2-vector space over K of rank n* $(n \geq 0)$ is a symmetric monoidal category $\mathbb{V}$ equipped with a (left) <u>action</u> of $\mathbf{Vect}_K$ and such that it is equivalent (as a $\mathbf{Vect}_K$-module category) to $\mathbf{Vect}_K^n$.

Example: G a finite group, K algebraically closed. Then, $\mathbf{Rep}_{\mathbf{Vect}_K}(G)$ is a 2-vector space of rank

$$n = \text{ number of conjugacy classes of } G$$

$$\left\{ \begin{array}{c} (0,\ldots,\overset{i)}{K},\ldots,0) \\ i=1,\ldots,n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{non equivalent} \\ \text{irreducible} \\ \text{representations} \\ \text{of } G \end{array} \right\}$$

$$\text{KV 2-vector spaces } = \left(\begin{array}{c} \text{objects in a} \\ \text{2-category } \mathsf{2Vect}_K^{\mathrm{KV}} \end{array} \right)$$

A simple model of $\mathsf{2Vect}_K^{\mathrm{KV}}$.

$\mathsf{2Vect}_\mathsf{K}^\mathsf{KV}$ is biequivalent to the 2-category $\mathsf{2Vect}_\mathsf{K}$ with

- objects: categories $\mathbf{Vect}_K^n$, $n \geq 0$
- 1-arrows: K-linear functors
- 2-arrows: natural transformations

Remark. There also exists a 2-category $\mathsf{2Mat}_K$ of *2-matrices over K* biequivalent to $\mathsf{2Vect}_\mathsf{K}$ (analog of $\mathbf{Mat}_K \simeq \mathbf{Vect}_K$).

4. Representation theory on $\mathsf{2Vect_K}$

A. Cohomological description of a representation

Put:

$$\boxed{\mathbb{GL}(n,K) \;\equiv\; \mathsf{Equiv}_{\mathsf{2Vect_K}}(\mathbf{Vect}_K^n)}$$

(General linear 2-group)

An object in $\mathsf{Rep}_{\mathsf{2Vect_K}}(\mathbb{G})$ is a pair $(\mathbf{Vect}_K^n, \mathbb{F})$ with

$$\mathbb{F} = (F, F_2) : \mathbb{G} \to \mathbb{GL}(n,K)$$

a morphism of 2-groups (n is called the $\mathsf{dimension\ of\ the\ representation}$).

Remark. For any $\mathbb{F}, \mathbb{F}' : \mathbb{G} \to \mathbb{GL}(n,K)$,

$$\underset{\text{(in }\mathsf{2Grps})}{\mathbb{F} \cong \mathbb{F}'} \quad \Longrightarrow \quad \underset{\text{(in }\mathsf{Rep}_{\mathfrak{C}}(\mathbb{G}))}{\mathbb{F} \simeq \mathbb{F}'}$$

Lemma. *Let $n \geq 0$. Then, $\mathbb{GL}(n, K)$ is split and*

$$\pi_0(\mathbb{GL}(n, K)) \cong \Sigma_n$$
$$\pi_1(\mathbb{GL}(n, K)) \cong (K^*)^n$$

with Σ_n acting on $(K^*)^n$ by

$$\sigma \cdot (\lambda_1, \ldots, \lambda_n) = (\lambda_{\sigma^{-1}(1)}, \ldots, \lambda_{\sigma^{-1}(n)})$$

Proposition. *Let $\mathbb{G}$ be any 2-group and α any classifying 3-cocycle of $\mathbb{G}$. Then, up to equivalence, a* linear representation *of $\mathbb{G}$ is given by a quadruple (n, ρ, β, c) with*

- *n a natural number ≥ 0*
- *$\rho : \pi_0(\mathbb{G}) \to \Sigma_n$ a group morphism*
- *$\beta : \pi_1(\mathbb{G}) \to (K^*)^n_\rho$ a morphism of $\pi_0(\mathbb{G})$-modules such that $[\beta\ \alpha] = 0$*
- *$c \in C^2(\pi_0(\mathbb{G}), (K^*)^n_\rho)$ a 2-cochain such that $\partial c = \beta\ \alpha$*

Example: $\dfrac{\text{1-dimensional linear representations}}{\text{(“characters” of } \mathbb{G})}$

Given by pairs (χ, c), with χ a $\pi_0(\mathbb{G})$-invariant character of $\pi_1(\mathbb{G})$ such that $[\chi\ \alpha] = 0$ and c as before.

In particular, there is a map

$$\begin{array}{ccc} Z^2(\pi_0(\mathbb{G}), K^*) & \longrightarrow & \{\text{“characters” of } \mathbb{G}\} \\ z & \mapsto & \mathcal{I}(z) \end{array}$$

$\mathcal{I}(z)$ defined by the pair $(\mathbf{Vect}_K, \mathbb{I}(z))$, with

$$\mathbb{I}(z) : \mathbb{G} \to \mathbb{GL}(1, K)$$

the *trivial constant functor* and non trivial monoidal structure given by

$$F_2(g_1, g_2)_V = z(g_1, g_2)\mathrm{id}_V$$

Rmk. These representations generalize to any dimension (*purely cocyclic representations*).

Theorem. (E, 2005) *There is a bijection*

$$\left\{\begin{array}{c}\text{Equivalence}\\ \text{classes of linear}\\ \text{representations}\\ \text{of } \mathbb{G}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Equivalence}\\ \text{classes of}\\ \text{quadruples}\\ (n,\rho,\beta,c)\end{array}\right\}$$

where $(n,\rho,\beta,c) \simeq (n',\rho',\beta',c')$ *iff*

- $n = n'$
- *there exists a permutation* $\sigma \in \Sigma_n$ *such that*

$$\begin{aligned} \rho'(g) &= \sigma\rho(g)\sigma^{-1} \\ \beta'(u) &= \sigma \cdot \beta(u) \\ [c'] &= [\sigma \cdot c] \end{aligned}$$

for all $g \in \pi_0(\mathbb{G})$ *and* $u \in \pi_1(\mathbb{G})$.

Example: Let D_{2m} be the dihedral group thought of as the split 2-group $\mathbb{Z}_2[0] * \mathbb{Z}_m[1]$. Then

$$\left\{\begin{array}{c}\text{eq. classes of}\\ \text{“characters”}\\ \text{of } D_{2m}\end{array}\right\} = \begin{cases} K^*, & \text{if } m = 2 \text{ or odd} \\ K^* \sqcup K^*, & \text{if } m \neq 2 \text{ even} \end{cases}$$

B. Categories of morphisms

• A geometric way of thinking of a pair of representations.

Given representations (n, ρ, β, c) and (n', ρ', β', c'), let

$$M(n', n) \equiv \{1, \ldots, n'\} \times \{1, \ldots, n\}$$

$$I(\beta, \beta') \equiv \{(i', i) \in M(n', n) \mid \beta'_{i'} = \beta_i\}$$

("intertwining points")

Then:

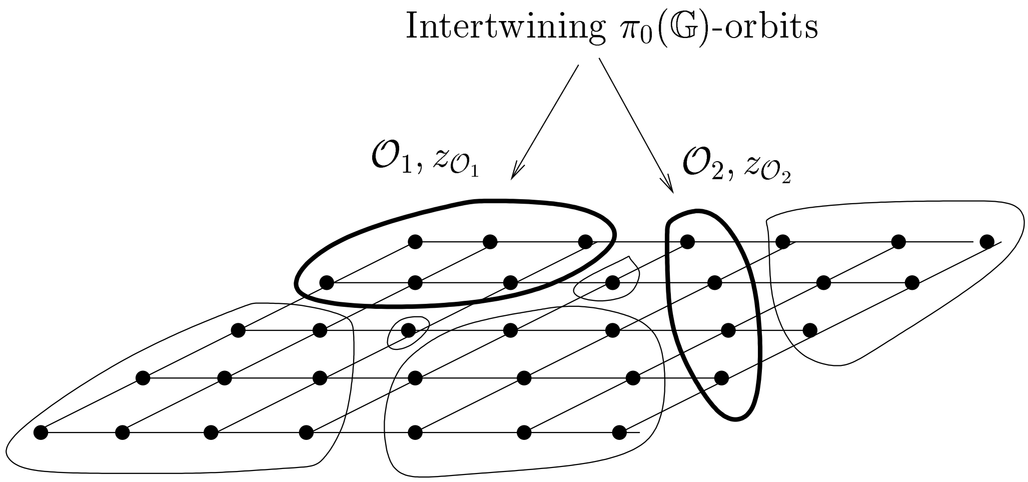
- $\pi_0(\mathbb{G})$ acts (on the right) on $M(n', n)$

$$(i', i) \cdot g = (\rho'(g)^{-1}(i'), \rho(g)^{-1}(i))$$

- If $(i', i) \in I(\beta, \beta')$, the remaining points in its $\pi_0(\mathbb{G})$-orbit are also intertwinig. Hence

$$I(\beta, \beta') = \bigsqcup_{\mathcal{O} \in \mathsf{Orb}_{\mathrm{Int}}} \mathcal{O}$$

Example: If $n = 7$ and $n' = 5$,



Geometric view of a pair of representations

$(7, \rho, \beta, c)$ and $(5, \rho', \beta', c')$

$$z_{\mathcal{O}} : \pi_0(\mathbb{G}) \times \pi_0(\mathbb{G}) \to \mathcal{F}(\mathcal{O}, K^*)$$

$$z_{\mathcal{O}}(g_1, g_2) \;\sim\; \frac{c'(g_1, g_2)}{c(g_1, g_2)}$$

• Geometric interpretation of a 1-morphism $(n, \rho, \beta, c) \to (n', \rho', \beta', c')$.

A 1-morphism $(n, \rho, \beta, c) \to (n', \rho', \beta', c')$ is given by a pair (H, Φ) with

- $H : \mathbf{Vect}_K^n \to \mathbf{Vect}_K^{n'}$ a K-linear functor
- Φ a collection of natural isomorphisms

$$\begin{tikzcd} \mathbf{Vect}_K^n \arrow[r, "F(A)"] \arrow[d, "H"'] \arrow[dr, phantom, "\overset{\Phi(A)}{\Leftrightarrow}"] & \mathbf{Vect}_K^{n'} \arrow[d, "H"] \\ \mathbf{Vect}_K^n \arrow[r, "F'(A)"'] & \mathbf{Vect}_K^{n'} \end{tikzcd}$$

A object in $\mathbb{G}$.

<u>What about H</u> ?

Up to isomorphism, H is given by a matrix $\mathsf{R} \in \mathrm{Mat}_{n' \times n}(\mathbb{N})$ (the <u>matrix of ranks</u>).

Lemma. *For any matrix* $\mathsf{R} \in \mathrm{Mat}_{n'\times n}(\mathbb{N})$, *let*

$$\mathrm{Sup}(\mathsf{R}) \equiv \{(i',i) \in M(n',n) \mid R_{i'i} \neq 0\}$$

Then, R *is the matrix of ranks for some 1-morphism* $(n,\rho,\beta,c) \to (n',\rho',\beta',c')$ *iff*

- R *is* (ρ',ρ)*-invariant* ($R_{\rho'(g)(i'),\rho(g)(i)} = R_{i'i}$)
- $\mathrm{Sup}(\mathsf{R}) \subseteq I(\beta',\beta)$

Let us think of R as

$$\begin{aligned} \mathsf{R} \ &\equiv \{p_{\mathcal{O}} : E_{\mathcal{O}} \to \mathcal{O}\}_{\mathcal{O}\in\mathsf{Orb}_{\mathrm{Int}}} \\ &p_{\mathcal{O}} : E_{\mathcal{O}} \to \mathcal{O} \text{ vector bundle} \\ &\mathsf{dim}_K E_{\mathcal{O}} = R_{i'i} \quad (i',i) \in \mathcal{O} \end{aligned}$$

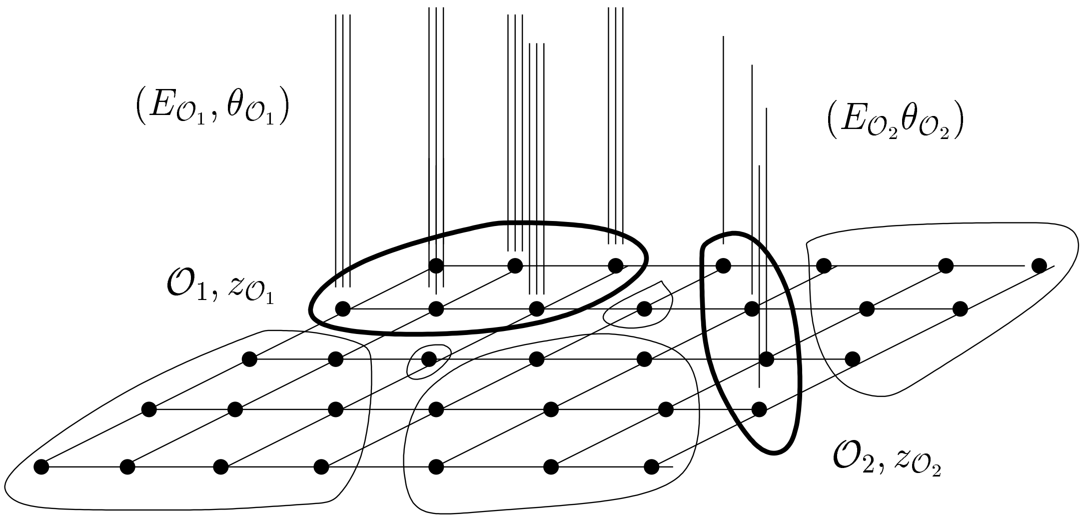
Hence

$$H \ \sim \ \text{collection of vector bundles } \{E_{\mathcal{O}}\}$$

<u>What about Φ</u> ?

Lemma. Φ is equivalent to a collection of projective right $\pi_0(\mathbb{G})$-actions $\{\theta_{\mathcal{O}},\ \mathcal{O} \in \mathsf{Orb}_{\mathrm{Int}}\}$ of cocycles $\{z_{\mathcal{O}}\}$, and covering the action of $\pi_0(\mathbb{G})$ on the orbits.

Example: If $n = 7$ and $n' = 5$,



Geometric interpretation of a 1-morphism

$(7, \rho, \beta, c) \to (5, \rho', \beta', c')$

• Geometric interpretation of 2-morphisms

$$\text{2-morphism} \longleftrightarrow \left(\begin{array}{c} \text{collection of morphisms} \\ \text{of vector bundles} \\ \text{preserving the actions} \\ \{f_{\mathcal{O}} : E_{\mathcal{O}} \to E'_{\mathcal{O}}\}_{\mathcal{O}} \end{array} \right)$$

Moreover, composition of 2-morphisms exactly corresponds to the composition of these $f_{\mathcal{O}}$.

Hence:

Theorem. (E, 2005) *For any* (n, ρ, β, c) *and* (n', ρ', β', c')*, there is an equivalence*

$$\mathbf{Hom}((n, \rho, \beta, c), (n', \rho', \beta', c')) \simeq$$
$$\simeq \prod_{\mathcal{O} \in \mathsf{Orb}_{\mathrm{Int}}} \mathbf{Bund}_{\pi_0(\mathbb{G}), z_{\mathcal{O}}}(\mathcal{O})$$

Example. For "characters" $\mathcal{I}(\chi, c)$, $\mathcal{I}(\chi', c')$

$$\mathbf{Hom}(\mathcal{I}(\chi, c), \mathcal{I}(\chi, c')) \simeq \begin{cases} \mathbf{1}, & \chi \neq \chi' \\ \mathbf{PRep}_{[c'-c]}(\pi_0(\mathbb{G})), & \chi = \chi' \end{cases}$$

with $\mathbf{PRep}_{[z]}(\pi_0(\mathbb{G}))$ the category of projective representations of $\pi_0(\mathbb{G})$ with cohomology class $[z] \in H^2(\pi_0(\mathbb{G}), K^*)$.

In particular, if $(\chi, c) = (\chi', c')$ are trivial

$$\boxed{\mathbf{End}_{\mathsf{Rep}_{\mathsf{2Vect_K}}(\mathbb{G})}(\mathcal{I}) \ \simeq \ \mathbf{Rep}_{\mathbf{Vect}_K}(\pi_0(\mathbb{G}))}$$

(equivalence of monoidal categories)

Rmk. $\mathsf{Rep}_{\mathsf{2Vect_K}}(\mathbb{G})$ is a *monoidal* 2-category and $\mathcal{I}$ is the unit object.

5. Reasons to study representations of 2-groups.

- It has been shown (Polesello-Waschkies, 2004) that representations of a 2-group $\mathbb{G}$ in $\mathfrak{C}$ may be identified with locally constant stacks on a suitable space X with values in $\mathfrak{C}$.
- If $\mathfrak{C}$ is monoidal, $\mathfrak{Rep}_{\mathfrak{C}}(\mathbb{G})$ inherits a monoidal structure, and it has been shown (Mackaay, 1999) that 4-manifold invariants can be built from certain monoidal 2-categories. Hence, interesting invariants of 4-manifolds may possibly be built from this monoidal 2-category of representations or from suitable deformations of it.
- Possible applications to theoretical physics...