Assignment 2

General Instructions: Due Feb. 15. Graduate students should attempt all problems. Undergraduate students should attempt at least 5 problems.

(1) Consider the planar vector field:

$$\boldsymbol{A} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y},$$

and the function $f(x, y) = xy$.

(a) Using directional derivatives show that $f(x, y)$ is a first integral.

(b) Use this fist integral to rectify $\boldsymbol{A}$.

(c) Determine the flow Φ generated by $\boldsymbol{A}$. Using an explicit calculation, verify that

$$(f \circ \Phi)(t, x, y) = f(x, y).$$

(2) Let $(r, \theta) = \boldsymbol{F}(x, y)$ be the transformation from Cartesian to polar coordinates. Let $\boldsymbol{A}$ be as in question 1, and let Φ be the flow generated by $\boldsymbol{A}$. Let $\boldsymbol{B} = \boldsymbol{F}_*\boldsymbol{A}$ and $\Psi = \boldsymbol{F}_*\Phi$ be the indicated pushforwards.

(a) Verify the principle of covariance: by direct calculation show that $\boldsymbol{B}$ generates Ψ.

(b) Use the principle of covariance to rectify the vector field

$$\boldsymbol{B} = x\cos(2y)\frac{\partial}{\partial x} - \sin(2y)\frac{\partial}{\partial y}.$$

(3) Let $g(x, y) = x^2 - y^2$, and let $(x, y) = \boldsymbol{G}(r, \theta)$ be the transformation from polar to Cartesian coordinates. Let $\boldsymbol{A}, \boldsymbol{B}, \Phi, \Psi$ be as above.

(a) Let $h = \boldsymbol{G}^* g$. By direct calculation, verify the principle of covariance for directional derivatives by showing that

$$\boldsymbol{B}[h](r, \theta) = \boldsymbol{A}[g](x, y).$$

(b) Let $u(t, x, y) = (\Phi_t^* g)(x, y)$ and $v(t, r, \theta) = (\Psi_t^* h)(r, \theta)$. Verify by explicit calculation that

$$u(t, x, y) = v(t, r, \theta).$$

(c) Finally, verify the geometric definition of the directional derivative by showing that

$$\boldsymbol{A}[g](x, y) = \dot{u}(0, x, y)$$
$$\boldsymbol{B}[h](r, \theta) = \dot{v}(0, r, \theta).$$

(4) Let (r, θ) be the usual polar coordinates. We showed in lecture that

$$\frac{x}{\sqrt{x^2 + y^2}}\frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2 + y^2}}\frac{\partial}{\partial y} = \frac{\partial}{\partial r}.$$

(a) Use the above rectification to determine the flow for the vector field $\partial/\partial r$.

(b) Verify that this flow gives the general solution for the ODE

$$\dot{x} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \dot{y} = \frac{y}{\sqrt{x^2 + y^2}}.$$

(5) Prove Proposition 5.4 of the class notes (coherence of pullback and push-forward).

(6) In lecture we showed that the vector field

$$\boldsymbol{A} = \frac{\partial}{\partial x} + (x + y)\frac{\partial}{\partial y}$$

generates the flow

$$\Phi(t, x, y) = (x + t, (x + y + 1)e^t - x - t - 1).$$

(a) Use the flow to determine a first integral of $\boldsymbol{A}$.

(b) Rectify $\boldsymbol{A}$.