Assignment 3

General Instructions: Due March 8 Graduate students should attempt all problems. Undergraduate students should attempt problems 1-5, 7,8. Problems 6,9 will count as extra credit.

(1) Consider the following planar vector fields:
$$\boldsymbol{A} = x\partial_x - y\partial_y, \qquad \boldsymbol{B} = -y\partial_x + x\partial_y.$$
  (a) Reexpress $\boldsymbol{A}, \boldsymbol{B}$ in polar coordinates.
  (b) Calculate the Lie bracket using Cartesian and polar coordinates, and verify the principle of covariance by showing that the two calculations agree.

(2) Let $\boldsymbol{A}, \boldsymbol{B}$ be the vector fields of question 1 and let Φ_t, Ψ_t be the corresponding flows.
  (a) Calculate the time dependent vector fields
$$\boldsymbol{C}_t = \Phi_{t*}\boldsymbol{B}, \qquad \boldsymbol{E}_t = \Psi_{t*}\boldsymbol{A}$$
  (b) Verify that
$$\dot{\boldsymbol{C}}_t = [\boldsymbol{C}_t, \boldsymbol{A}]$$
$$\dot{\boldsymbol{E}}_t = [\boldsymbol{E}_t, \boldsymbol{B}].$$
  (c) Finally, verify that
$$\dot{\boldsymbol{C}}_0 = [\boldsymbol{B}, \boldsymbol{A}]$$
$$\dot{\boldsymbol{E}}_0 = [\boldsymbol{A}, \boldsymbol{B}].$$

(3) (a) Find the most general infinitesimal symmetry of ∂_x. In other words, describe all planar vector fields $\boldsymbol{A}$ such that
$$[\boldsymbol{A}, \partial_x] = 0.$$
  (b) Find the most general symmetry of ∂_x. In other words, find the most general transformation $(u, v) = \boldsymbol{F}(x, y)$ such that $\boldsymbol{F}_*\partial_x = \partial_x$, or what is equivalent such that $\partial_x = \partial_u$.
  (c) Find the most general infinitesimal conformal symmetry of ∂_x. In other words, describe all planar vector fields $\boldsymbol{A}$ such that
$$[\boldsymbol{A}, \partial_x] = f(x, y)\partial_x.$$
  (d) Find the most general form of an ODE $dy/dx = \omega(x, y)$ for which ∂_x is an infinitesimal symmetry.

(4) (a) What is the most general vector field $\boldsymbol{A}$ for which $-y\partial_x + x\partial_y$ is an infinitesimal symmetry? In other words, describe all planar vector fields $\boldsymbol{A}$ such that
$$[-y\partial_x + x\partial_y, \boldsymbol{A}] = 0.$$
  Hint: switch to polar coordindates and utilize the principle of covariance.

(b) What is the most general vector field $\boldsymbol{A}$ for which $-y\partial_x + x\partial_y$ is an infinitesimal conformal symmetry? In other words, describe all planar vector fields $\boldsymbol{A}$ such that $[-y\partial_x + x\partial_y, \boldsymbol{A}]$ is proportional to $\boldsymbol{A}$.

(5) Prove that the Lie bracket obeys the Jacobi identity.

(6) Prove the following Proposition. Let $\boldsymbol{F} : U \to V$ be a diffeomorphism and $\boldsymbol{G} : V \to U$ the inverse transformation. For $\boldsymbol{A} : U \to \mathbb{R}^n$, a vector field, and $f : U \to \mathbb{R}$, a function

$$\boldsymbol{F}_*(g\boldsymbol{B}) = \boldsymbol{G}^*(g)\boldsymbol{F}_*(\boldsymbol{B}). \tag{1}$$

(7) A Bernoulli equation is a first-order ODE having the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \quad n = 1, 2, \ldots$$

(a) Consult a standard ODE text and describe the usual "cookbook" procedure for integrating a Bernoulli equation.

(b) Show that a Bernoulli equation admits a symmetry of the form

$$f(x)y^n\partial_y.$$

Recover the standard integration method by using the symmetry.

(8) (a) Describe the most general first-order ODE that admits $-y\partial_x + x\partial_y$ as a symmetry. Describe the integration procedure for such equations. Hint: see question 4.

(b) Use this method to integrate

$$\frac{dy}{dx} = \frac{x^3 + xy^2 + y}{x - y^3 - x^2y} = \frac{x(x^2 + y^2) + y}{x - y(x^2 + y^2)}.$$

(9) (a) Let $\boldsymbol{A} = g(x)(x\partial_x + y\partial_y)$. Show that

$$\frac{dy}{dx} = y/x + \frac{f(y/x)}{g(x)}$$

is the most general ODE that admits $\boldsymbol{A}$ as an infinitesimal symmetry. Hint: review the derivation of symmetries for homogeneous first-order ODEs.

(b) Describe the corresponding integration method.

(c) Use the method to integrate the ODE

$$\frac{dy}{dx} = y/x + e^{y/x}\ln(x).$$