Assignment 4

General Instructions: Due March 22. Graduate students should attempt all problems. Undergraduate students should attempt problems 1,3,4,6. Problems 2,5 will count as extra credit.

(1) (a) Let $U, V, W \subset \mathbb{R}^2$ be 2-dimensional domains and $\boldsymbol{F} : U \to V$ and $\boldsymbol{G} : V \to W$ diffeomorphisms. Prove that

$$(\boldsymbol{G} \circ \boldsymbol{F})^{(1)} = \boldsymbol{G}^{(1)} \circ \boldsymbol{F}^{(1)}$$

Hint: note that for $(u, v) = \boldsymbol{F}(x, y)$ the transformation of the derivative

$$v_1 = \frac{v_x + v_y y_1}{u_x + u_y y_1}$$

is a fractional linear transformation.

(b) Let $\boldsymbol{A} = \xi(x, y)\partial_x + \eta(x, y)\partial_y$ be a C^2 vector-field defined on a 2-dimensional domain with coordinates (x, y). Let Φ_t be the C^2 flow generated by $\boldsymbol{A}$. Prove that the 1-parameter family of prolonged transformations $\Phi_t^{(1)}$ defines a flow.

(c) Prove that the prolonged vector field $\boldsymbol{A}^{(1)}$ generates the prolonged flow $\Phi_t^{(1)}$. Hint: first, write $(u, v) = \Phi_t(x, y)$ and prove that

$$\left.\begin{pmatrix} \dot{u}_x & \dot{u}_y \\ \dot{v}_x & \dot{v}_y \end{pmatrix}\right|_{t=0} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}.$$

(2) Let $\boldsymbol{A} = \xi(x, y)\partial_x + \eta(x, y)\partial_y$ be a vector field on a 2-dimensional domain with coordinates (x, y). Let

$$\boldsymbol{A}^{(1)} = \xi(x, y)\partial_x + \eta(x, y)\partial_y + \nu(x, y, y_1)\partial_{y_1}$$

where

$$\nu(x, y, y_1) = (\partial_x + y_1\partial_y)[\eta(x, y) - y_1\xi(x, y)]$$

be the prolongation of $\boldsymbol{A}$, a vector field in the 3-dimensional domain with coordinates (x, y, y_1). Let S_ω be the surface in this 3-dimensional domain defined by the equation $y_1 = \omega(x, y)$. Prove that $\boldsymbol{A}$ is an infinitesimal symmetry of a 1st-order ODE

$$\frac{dy}{dx} = \omega(x, y)$$

if and only if $\boldsymbol{A}^{(1)}$ is tangent to the surface S_ω. Hint: review the symmetry determining equation.

(3) (a) Show that ∂_y is an infinitesimal symmetry of the second-order ODE having the form

$$\frac{d^2y}{dx^2} = \omega\left(x, \frac{dy}{dx}\right). \tag{1}$$

(b) Show that the above ODE admits the following reduction of order:

$$\frac{dy_1}{dx} = \omega(x, y_1), \quad \frac{dy}{dx} = y_1.$$

(4) Let $y = f(x)$ be a non-zero solution of the second-order homogeneous linear ODE

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0. \tag{2}$$

(a) Consult a standard ODE text and describe the usual procedure for reduction of order in such situations.
(b) Show that $f(x)\partial_y$ is a symmetry of the above ODE.
(c) Show that $f(x)\partial_y = \partial_v$ where $u = x,\ v = y/f(x)$, and that correspondingly the ODE in question takes the form

$$\frac{d^2v}{du^2} = \omega\left(u, \frac{dv}{du}\right).$$

Conclude by showing that the reduction of order described in question 6 is equivalent to the standard reduction of order procedure described in part (a) of the present question.

(5) A hodograph transformation is a change of coordinates that reverses the roles of the dependent and independent variables. For scalar ODEs the hodograph transformation is simply

$$\hat{x} = y, \quad \hat{y} = x.$$

In particular we say that functions $y = f(x)$ and $\hat{y} = \hat{f}(\hat{x})$ are related by a hodograph transformation if

$$x = \hat{y} = \hat{f}(\hat{x}) = \hat{f}(y) = \hat{f}(f(x));$$

in other words if f and $\hat{f}$ are functional inverses.

(a) Show that second-order ODEs

$$\frac{d^2y}{dx^2} = \omega\left(x, y, \frac{dy}{dx}\right), \quad \frac{d^2\hat{y}}{d\hat{x}^2} = \hat{\omega}\left(\hat{x}, \hat{y}, \frac{d\hat{y}}{d\hat{x}}\right)$$

are related by a hodograph transformation if and only if

$$\hat{\omega}(\hat{x}, \hat{y}, \hat{y}_1) = -(\hat{y}_1)^3\,\omega(\hat{y}, \hat{x}, 1/\hat{y}_1)$$

(b) Use the hodograph transformation to solve the ODE

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^3.$$

(c) Use the above transformation law to show that the hodograph transformation relates an ODE of the form

$$\frac{d^2y}{dx^2} = \omega\left(x, \frac{dy}{dx}\right) \tag{3}$$

(see Question 6) to an autonomous ODE.

(6) (a) Let y be a real number, and consider the function $\hat{T}_y : \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{T}_y : x \mapsto x + (y - \sin(x))\,, \quad x \in \mathbb{R}.$$

Fix a value of y between -1 and 1 (your choice), set $x_0 = 0$, and inductively define $x_{k+1} = \hat{T}_y(x_k)$. Write down the first few elements of the x_k sequence. Compare these numbers to $\arcsin(y)$. Report your findings. Next, fix a value of $y > 1$, and again write down the first few elements of the sequence x_k. Report your findings.

(b) Let $\mathcal{C}_\epsilon = \mathcal{C}([-\epsilon, \epsilon], \mathbb{R})$ and consider the operator $T : \mathcal{C}_\epsilon \to \mathcal{C}_\epsilon$ defined by

$$T[g](y) = g(y) + (y - \sin(g(y))), \quad g \in \mathcal{C}_\epsilon\,.$$

Let $g_0 = 0 \in \mathcal{C}_\epsilon$ be the zero function and inductively define $g_{k+1} = T[g_k]$. Write out the first few functions in the sequence $\{g_k\}_{k=0}^{\infty}$

(c) Use Maple to plot the first few elements of the sequence $g_k(y)$ over the range $-1.5 \le y \le 1.5$. Describe the apparent convergence properties. Use Maple to plot the difference $g_k(y) - \arcsin(y)$ over the range $-1 \le y \le 1$ and, again, describe the apparent convergence properties.

(d) Let K be a real number strictly between 0 and 1. Set $\delta = \sqrt{2K}$. Show that for $x_1, x_2 \in B_\delta(0)$ and for every y we have

$$|\hat{T}_y(x_1) - \hat{T}_y(x_2)| \le K|x_1 - x_2|.$$

Hint: use the mean value theorem and the fact (explain) that

$$\frac{x^2}{2} > 1 - \cos(x).$$

(e) Set $\epsilon = \delta(1 - K)$. Let $x \in B_\delta(0)$ and $y \in B_\epsilon(0)$, and set $\hat{x} = \hat{T}_y(x)$. Prove that $\hat{x} \in B_\delta(0)$ also. Hint, you will need to show that

$$|x - \sin(x)| \le K|x|.$$

(f) Fix $\delta > 0$ and let $\mathcal{C}_{\epsilon,\delta} \subset \mathcal{C}_\epsilon$ be the closed subset consisting of functions whose absolute value is bounded by δ, i.e., the closed ball of radius δ around the zero function in $\mathcal{C}_\epsilon$. With T as above, show that $T : \mathcal{C}_{\epsilon,\delta} \to \mathcal{C}_{\epsilon,\delta}$ is a well defined operator Hint: use part (e).

(g) Show that $T : \mathcal{C}_{\epsilon,\delta} \to \mathcal{C}_{\epsilon,\delta}$ is a contraction operator with Lipschitz constant K. Hint: use part (d).

(h) Use the fixed point theorem to conclude that the function $g(y) = \arcsin(y),\ |y| \le \epsilon$ is the unique fixed point of the operator $T : \mathcal{C}_{\epsilon,\delta} \to \mathcal{C}_{\epsilon,\delta}$