Assignment 5

General Instructions: Due April 7. Graduate students should attempt all problems. Undergraduate students should attempt problems 2,3,4. Problems 1,5 will count as extra credit.

(1) Let A be an $n \times n$ real matrix and

$$\|A\| = \sup \left\{ \frac{\|Au\|}{\|u\|} : u \in \mathbb{R}^n, u \neq 0 \right\}$$

the operator norm, relative to the Euclidean norm

$$\|u\| = \sqrt{u \cdot u}, \quad u \in \mathbb{R}^n.$$

(a) Prove that

$$\begin{aligned} &\|A + B\| \leq \|A\| + \|B\|, \quad A, B \in \mathrm{Mat}_{n \times n} \mathbb{R} \\ &\|AB\| \leq \|A\| \, \|B\|. \end{aligned}$$

(b) Let $S = S^t$ be a symmetric matrix. Prove that

$$\|S\| = \max\{|\lambda| : \det(S - \lambda I) = 0\}$$

(2) (a) Find the Jordan canonical form J of the matrix

$$A = \begin{pmatrix} -1 & 3 & 0 \\ -3 & -1 & 3 \\ 0 & 3 & -1 \end{pmatrix}.$$

(b) Calculate $\exp(tJ)$.

(c) What is the general solution of the linear ODE

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}.$$

(3) For a complex number $z = a + \mathrm{i}\, b$, let

$$z^{\mathrm{R}} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

If A is an $n \times n$ complex matrix, let A^{R} denote the $(2n) \times (2n)$ matrix obtained by replacing each entry $A_{ij} \in \mathbb{C}$ with A^{R}_{ij}.

(a) Let A, B be $n \times n$ complex matrices. Prove that $(AB)^{\mathrm{R}} = A^{\mathrm{R}} B^{\mathrm{R}}$

(b) Let $a, b \in \mathbb{R}$. Calculate $\exp(tB)$, where

$$B = \begin{pmatrix} a & -b & 1 & 0 \\ b & a & 0 & 1 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{pmatrix}.$$

Hint: find a 2×2 complex matrix A such that $B = A^{\mathrm{R}}$.

(4) In this exercise we will use Picard iteration to obtain the solution to the linear ODE:

$$\dot{x} = -y, \quad \dot{y} = x.$$

(a) Let $\mathbf{V} : \mathbb{R}^2 \to \mathbb{R}^2$ denote the above vector field function, i.e.

$$\mathbf{V}(x, y) = (-y, x).$$

What is the flow generated by $\mathbf{V}$?

(b) Let $D = B_1(0) \subset \mathbb{R}^2$ denote the closed unit disk in $\mathbb{R}^2$, and let $I = [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}$. Let

$$\mathcal{F} = \{\phi \in \mathcal{C}^0(I \times D, \mathbb{R}^2) : \phi(0, x, y) = (x, y)\}.$$

Define the operator $\mathcal{P} : \mathcal{F} \to \mathcal{F}$ by

$$\mathcal{P}[\phi](t, x, y) = (x, y) + \int_0^t \mathbf{V}(\phi(s, x, y))ds, \quad \phi \in \mathcal{F}$$

Define $\phi_0(t, x, y) = (x, y)$, and then $\phi_{k+1} = \mathcal{P}[\phi_k]$. Explicitly determine $\phi_1, \phi_2, \phi_3, \phi_4$.

(c) Prove that the sequence of functions $\{\phi_k\}_{k=0}^{\infty}$ converges uniformly to the flow determined in part (a). Hint: it's possible to write $\phi_k(t, x, y)$ using the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and its various powers. As well, in proving convergence, it may be useful to identify $\mathbb{R}^2$ with $\mathbb{C}$ and to re-express the matrix A as the imaginary number i.

(d) Prove that $\mathbf{V} : \mathbb{R}^2 \to \mathbb{R}^2$ is a Lipschitz function with Lipschitz constant 1.

(e) Prove explicitly that $\mathcal{P}$ is a contraction operator and that ϕ, the flow generated by $\mathbf{V}$ is its unique fixed point.

(5) (a) Find the general solution of the time-dependent linear ODE

$$\dot{x} = -\cos(t)y, \quad \dot{y} = \cos(t)x.$$

Hint: decouple the equations.

(b) Autonomize the above ODE and give the flow generated by the corresponding 3-dimensional vector field.