# Math 4190/5190 Final Exam — Winter 2011

**General Instructions:** Due April 14. Graduate students should attempt all problems. Undergraduates should complete 6 or more problems (extra problems give extra credit). Show all work. Explain all calculations. Justify all solutions. Barring exceptional circumstances, late submissions will not be accepted. Complete as much as you can by the deadline.

**The honour system.** This is a take-home examination. As such, it must be understood that this examination is to be a **STRICTLY** individual effort. You may not discuss the problems with other students, with tutors, with faculty members, with anyone. If you have a question about a problem, contact the instructor, but please be aware that I will not provide hints, only clarifications for any potentials ambiguities. Other than that, you may use all and any resources to complete these questions. The use of Maple is particularly encouraged. To ensure the fairness and integrity of this examination, I ask everyone to sign the following pledge:

I ____________________________ (name and signature) pledge to participate in this examination in a strictly individual fashion, without external assistance of any kind.

**Examination questions:**

(1) Let $A = x\partial_x + 3y\partial_y$
  (a) Find a 1st integral of $A$.
  (b) Rectify $A$.

(2) (a) Let $A$ be the vector field in question 1. Show that $A$ is a symmetry of $dy/dx = x^2 f(y/x^3)$
  (b) Conversely, show that if a 1st order ODE admits $A$ as a symmetry, it must have the form shown above.

(3) Find the general solution of the ODE

$$\frac{dy}{dx} = \frac{2y}{x} + x\cos\left(\frac{y}{x^2}\right).$$

(4) Let $A = x\partial_x + 2y\partial_y$
  (a) Determine $A^{(1)}$, the 1st prolongation of $A$.
  (b) Show that $A$ is a symmetry of the following 2nd order ODE:

$$\frac{d^2y}{dx^2} = f\left(\frac{y}{x^2}, \frac{dy/dx}{x}\right).$$

(5) Find the general solution of

$$\frac{d^2y}{dx^2} = \left(\frac{dy/dx}{x} - \frac{2y}{x^2}\right)^2 + 3\frac{dy/dx}{x} - \frac{4y}{x^2}.$$

(6) (a) Use Picard iteration to find the 1st and 2nd order approximations to the solution of the following Lotka-Volterra equation:

$$\frac{dx}{dt} = x - xy, \quad \frac{dy}{dt} = xy - y, \quad x(0) = y(0) = 2.$$

  (b) Use Maple to solve the above equation numerically. Then do a parametric plot comparing the approximate solution in part (a) to the numerical solutions for the range $-1 \le t \le 1$.

(7) (a) Find the general solution of the following ODE:

$$\frac{dx}{dt} = -11x + 15y$$
$$\frac{dy}{dt} = 15x - 6y$$

(b) Find a strict Lyapunov function for the above ODE. Justify your choice. Hint: diagonalize and use the appropriate quadratic function.

(8) (a) Find the Jordan canonical form of the following nilpotent matrix:

$$N = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}$$

(b) Calculate $\exp(tN)$.

(9) Consider the autonomous differential equation for the motion of a pendulum with friction:

$$x''(t) = -\sin(x(t)) - kx'(t),$$

where the parameter $k \geq 0$ plays the role of the coefficient of friction.

(a) Re-express the above equation as a first order ODE in two variables, i.e., as a vector field in the plane. What are the singular points of this vector field, and the corresponding linearizations?

(b) Determine the topological signature of each singular point, and classify each as a sink (stable), a source (unstable), or a saddle (unstable). How does your answer depend on the value of $k$?

(c) Use reduction of order to re-express the given equation as a 1st-order, but non-autonomous ODE. Hint: treat velocity as a function of position.

(d) Letting $U(x)$ denote the potential energy of the system, let $E(t) = \frac{1}{2}(x'(t))^2 + U(x(t))$ denote the total energy. What is $U(x)$? Calculate $E'(t)$. Use your calculation to explain how the presence of the friction parameter $k$ make the system non-conservative.

(e) Use Maple to create a phase plot of the give ODE for various values of $k$. Superimpose your plots with the level curves of the energy as a function of position $x$ and velocity $x'$. Discuss.

Here is some Maple code for generating phase plots of the conservative pendulum system.

```
with(plots):
with(DEtools):

deq:=[diff(x(t),t,t) = -sin(x(t)),x(0)=0,D(x)(0)=x1]:
T2:=64:
T1:=-2:
x1delta:=0.5:
x1max:=6:
x1min:=-2:
x1seq:=seq(i*x1delta,i= floor(x1min/x1delta) .. floor(x1max/x1delta)):
intcurveplots:=
       seq(odeplot(dsolve(deq,x(t),numeric,
```

```
                    range=(T1/(abs(x1)+1))..T2/(abs(x1)+1)
                   ),
               [x(t),D(x)(t)],refine=1
             ),
        x1=[x1seq] ):

    display(intcurveplots);
```

In the above approach we ask Maple to solve the 2nd order equation, and then use the solution to create a 2-dimensional parameterized plot (the `[x(t),D(x)(t)]` code fragment). To get multiple integral curves we begin with the initial condition $x(0) = 0$ and vary the initial condition $x'(0) = \text{const.}$ (this is the role of the x1seq variable). The odd looking range command is there to produce integral curves of comparable length.

The other way to get such plots is to use the reduction of order "trick" and plot $x'$ directly as a function of $x$. Here is a code snippet for doing just that:

```
autdeq:=[D(v)(x) = -sin(x)/v(x),v(0)=x1]:
xmax:=12:
xmin:=-12:
x1delta:=0.5:
x1max:=6:
x1min:=-6:
x1seq:=seq(i*x1delta,i= floor(x1min/x1delta) .. floor(x1max/x1delta)):

intcurveplots:=
    seq(odeplot(dsolve(autdeq,v(x),numeric,range=xmin..xmax)),x1=[x1seq]):
display(intcurveplots);
```

Note: the above commands when executed in Maple produce a slew of warnings. Why? Also note that the DEplot command can be used for this exercise. However, this will give you a direction field, rather than a true phase-plot. The visuals are quite similar, though.