# Jordan canonical form

## A bit of theory

Let N be an n×n nilpotent matrix of order d.
For us this means that $N^d \neq 0$ but that $N^{d+1} == 0$

**Theorem:** N is similar to a Jordan−canonical matrix having
$m_j$ Jordan blocks of size j×j such that
$m_1 + 2\, m_2 + 3\, m_3 + \ldots + d\, m_d = n$

The above theorem has a useful refinement.
Let $K_j = \ker(N^j)$ and $R_j = \mathrm{image}(N^j)$
and let $n_j = \dim K_j$ and $r_j = \dim R_j$ be the
corresponding nullities and ranks.
By the rank−nullity theorem, we have $n_j + r_j = n$

**Proposition:** $K_j \subset K_{j+1}$ and $R_{j+1} \subset R_j$ where the inclusions are strict for j=0,1,…, d
The hard part is proving that the inclusions are proper
By assumption, there exists a vector u such that $N^d\, u \neq 0$
Set $u_j = N^j\, u$ and observe that $N^j\, u_{d-j} \neq 0$ while $N^{j+1}\, u_{d-j} == 0$

**Proposition:** the following relations hold:
$r_{d-1} = m_d$
$r_{d-2} = 2\, m_d + m_{d-1}$
$r_{d-3} = 3\, m_d + 2\, m_{d-1} + m_{d-2}$
etc
The idea is that we pick vectors $u^{(d)}{}_i$, $i = 1 \ldots m_d$ that are independent modulo $K_d$
We then set $u^{(d)}{}_{j,i} = N^j\, u^{(d)}{}_i$
All these vectors span a $d \times m_d$ dimensional subspace that corresponds to $m_d$ Jordan blocks of size d×d
We then pick vectors $u^{(d-1)}{}_i$, $i = 1 \ldots m_{d-1}$ that belong to $K_d$ but that are independent modulo $K_{d-1}$ and modulo
$u^{(d)}{}_{1,i}$. These vectors then generate the $m_{d-1}$ Jordan blocks of size (j−1)×(j−1)
Continuing in like fashion we obtain a basis relative to which N has Jordan canonical form

## A 3+2+1 example

The idea here is to come up with a 6×6 nilpotent matrix
with a blocks of size 3,2,1.
We begin with a matrix in JCF

In[26]:=
```
N1 :=
  {{ 0, 0, 0, 0, 0, 0},
   { 0 , 0, 1, 0, 0, 0},
   { 0, 0, 0, 0, 0, 0},
   {0, 0, 0, 0, 1, 0},
   {0, 0, 0, 0, 0, 1},
   {0, 0, 0, 0, 0, 0}};

N1 // MatrixForm
N1. N1 // MatrixForm
N1.N1.N1 // MatrixForm
```

Out[27]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[28]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[29]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

And now we scramble it relative to a randomly chosen basis
We just have to make sure that the determinant is ≠0 (very small odds of a singular matrix being chosen, but it’s not impossible)

In[10]:=
```
P = Array[Random[Integer, {0, 1}] &, {6, 6}]
P // MatrixForm
Det[P]
```

Out[10]= {{0, 0, 0, 1, 0, 0}, {1, 0, 1, 1, 0, 1}, {0, 1, 1, 1, 0, 1}, {0, 0, 1, 1, 1, 0}, {0, 1, 1, 1, 1, 0}, {1, 1, 0, 0, 0, 1}}

Out[11]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[12]= -1

Now we scramble our initial matrix to obtain a nilpotent matrix
which is not in JCF
All the eigenvalues are zero, of course

In[43]:=
```
N2 = Inverse[P] . N1 . P;
N2 // HoldForm // # → (# // ReleaseHold // MatrixForm) &
N2.N2 // HoldForm // # → (# // ReleaseHold // MatrixForm) &
N2.N2.N2 // HoldForm // # → (# // ReleaseHold // MatrixForm) &
CharacteristicPolynomial[N2, x] // HoldForm // # → (# // ReleaseHold) &
```

Out[44]= $N2 \to \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 2 \\ 1 & 0 & -1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 2 & -2 \\ -2 & -1 & 1 & 1 & 2 & -3 \end{pmatrix}$

Out[45]= $N2.N2 \to \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 2 \\ 2 & 2 & 0 & 0 & 0 & 2 \end{pmatrix}$

Out[46]= $N2.N2.N2 \to \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Out[47]= CharacteristicPolynomial[N2, x] → $x^6$

Lets find the Jordan canonical form of N2
What are the possibilities:
6=3+2+1
6=3+3
6=3+1+1+1
The calculations below give us
m3 = r2 = 1
m2 = r1−2×m2 = 3−2=1
m1 = r0−3×m2−2×m1 = 6−3−2=1

In[73]:=
```
n2 → (NullSpace[N2.N2] // Length)
n1 → (NullSpace[N2] // Length)
r2 → 6 - (NullSpace[N2.N2] // Length)
r1 → 6 - (NullSpace[N2] // Length)
```

Out[73]= n2 → 5

Out[74]= n1 → 3

Out[75]= r2 → 1

Of course, by construction, its 3+2+1
This means that we can find a basis w3,w2,w1, v2, v1, u1
such that
w2=N.w3
w1=N.w2
v1=N.v2

Relative to such a basis, our nilpotent matrix will assume JCF
We start by looking for a vector w3 st that $N2^2$. w3 ≠0
Look at N2^2 and pick something not in its kernel

In[63]:=
```
NullSpace[N2.N2] // HoldForm //
 # → (# // ReleaseHold // RowReduce // Transpose // MatrixForm) &
w3 = {1, 0, 0, 0, 0, 0}
w2 = N2.w3
w1 = N2.w2
N2.w1
```

Out[63]= $\text{NullSpace}[N2.N2] \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 \end{pmatrix}$

Out[64]= {1, 0, 0, 0, 0, 0}

Out[65]= {1, 1, 1, 0, -1, -2}

Out[66]= {-1, -1, -1, 0, 2, 2}

Out[67]= {0, 0, 0, 0, 0, 0}

Since there is only one 3×3 block are next task
is to look for a u2 such that
$N^2$ u2 == 0 but N*u2 ≠ 0 and such that u2, w2 are independent

In[141]:=
```
"w2" → ({w2} // Transpose // MatrixForm)
NullSpace[N2.N2] // Transpose // MatrixForm
v2 = NullSpace[N2.N2][[1]]
v1 = N2. v2
N2.v1
```

Out[141]= $\text{w2} \to \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ -1 \\ -2 \end{pmatrix}$

Out[142]//MatrixForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[143]= {-1, 0, 0, 0, 0, 1}

Out[144]= {1, 0, 1, 0, -1, -1}

Out[145]= {0, 0, 0, 0, 0, 0}

So this just leaves a 1x1 block
choose a u1 at random from Ker(N2)
but make sure it's independent from w1,v1
Our first choice doesn't work out, so we pick another basis element of ker(N2)

In[160]:=
```
"w1" → ({w1} // Transpose // MatrixForm)
"v1" → ({v1} // Transpose // MatrixForm)
NullSpace[N2] // Transpose // MatrixForm
u1 = NullSpace[N2][[1]]
{w1, v1, u1} // RowReduce

u1 = NullSpace[N2][[2]]
{w1, v1, u1} // RowReduce
```

Out[160]= $w1 \to \begin{pmatrix} -1 \\ -1 \\ -1 \\ 0 \\ 2 \\ 2 \end{pmatrix}$

Out[161]= $v1 \to \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \\ -1 \end{pmatrix}$

Out[162]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Out[163]= {0, -1, 0, 0, 1, 1}

Out[164]= {{1, 0, 1, 0, -1, -1}, {0, 1, 0, 0, -1, -1}, {0, 0, 0, 0, 0, 0}}

Out[165]= {1, -1, 0, 1, 0, 0}

Out[166]= {{1, 0, 0, 1, -1, -1}, {0, 1, 0, 0, -1, -1}, {0, 0, 1, -1, 0, 0}}

Now we have the desired basis
Use this basis to form a change of basis matrix and conjugate
Voila! we have a matrix in JCF
Note: the basis we constructed isn't the randomly chosen basis we started from.
There are lots of ways to put a nilpotent matrix into JCF!

In[167]:=
```
P1 = {u1, v1, v2, w1, w2, w3} // Transpose;
P1 // MatrixForm
Inverse[P1].N2.P1 // MatrixForm
```

Out[168]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & -1 & 1 & 2 & -2 & 0 \end{pmatrix}$$

Out[169]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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