Math 4190/5190 — Solutions to assignment 1

(1) Let $\Phi(t,\mathbf{x})$ be a flow in n-dimensional space, and let $\mathbf{V}(\mathbf{x}) = \dot{\Phi}(0,\mathbf{x})$ be the corresponding vector field. Recall that $\dot{\Phi}$ is the derivative with respect to the first "time" variable, and that for a fixed t, we use Φ_t to denote the function $\mathbf{x} \mapsto \Phi(t,\mathbf{x})$.

(a) Use the chain rule to prove the following identities:

$$\dot{\Phi}_t = \mathcal{J}\Phi_t \cdot \mathbf{V} \tag{1}$$

$$\dot{\Phi}_t = \mathbf{V} \circ \Phi_t. \tag{2}$$

Conclude that for all t,

$$(\Phi_t)_*\mathbf{V} = \mathbf{V}. \tag{3}$$

Note: in the first equation the $\cdot$ symbol denotes matrix-vector multiplication. In the second equation the $\circ$ symbol denotes composition.

Solution: Since Φ is a flow, we have

$$\Phi(t,\Phi(s,\mathbf{x})) = \Phi(t+s,\mathbf{x}).$$

Taking the partial derivative with respect to s and using the chain rule gives

$$\frac{\partial\Phi(t+s,\mathbf{x})}{\partial s} = \dot{\Phi}(t+s,\mathbf{x}),$$

$$\frac{\partial\Phi(t,\Phi(s,\mathbf{x}))}{\partial s} = \mathcal{J}\Phi_t(\Phi(s,\mathbf{x}))\cdot\dot{\Phi}(s,\mathbf{x}).$$

Setting $s=0$ and using the fact that $\Phi(0,\mathbf{x}) = \mathbf{x}$ and that $\dot{\Phi}(0,\mathbf{x}) = \mathbf{V}(\mathbf{x})$ gives

$$\dot{\Phi}(t,\mathbf{x}) = \mathcal{J}\Phi_t(\mathbf{x})\cdot\mathbf{V}(\mathbf{x}).$$

This proves relation (1). Next, we take the partial derivative with respect to t to obtain

$$\frac{\partial\Phi(t+s,\mathbf{x})}{\partial t} = \dot{\Phi}(t+s,\mathbf{x}),$$

$$\frac{\partial\Phi(t,\Phi(s,\mathbf{x}))}{\partial t} = \dot{\Phi}(t,\Phi(s,\mathbf{x})).$$

Setting $t=0$ gives

$$\dot{\Phi}(s,\mathbf{x}) = \mathbf{V}(\Phi(s,\mathbf{x})),$$

which is a restatement of (2). Since the inverse of Φ_t is the transformation Φ_{-t}, by definition of push-forward

$$(\Phi_t)_*\mathbf{V} = (\mathcal{J}\Phi_t\cdot\mathbf{V})\circ\Phi_{-t}.$$

Hence, by (1) and (2),

$$(\Phi_t)_*\mathbf{V} = \dot{\Phi}_t\circ\Phi_{-t} = (\mathbf{V}\circ\Phi_t)\circ\Phi_{-t} = \mathbf{V}.$$

(b) Verify that the following transformations are flows. Calculate the corresponding vector fields. In both cases, verify that the flow gives the solutions of the corresponding ODE.

(i) $\Phi(t, x, y) = (x + t, y - f(x) + f(x + t))$, where $f(x)$ is fixed C^1 function.

SOLUTION: Evidently $\Phi(0, x, y) = (x, y)$. Let us check the 1-parameter group law. We have

$$\begin{aligned}
&\Phi(s, \Phi(t, x, y)) = \Phi(s, x + t, y - f(x) + f(x + t)) \\
&= (x + s + t, y - f(x) + f(x + t) - f(x + t) + f(x + s + t)) \\
&= (x + s + t, y - f(x) + f(x + s + t)) \\
&= \Phi(s + t, x, y)
\end{aligned}$$

The corresponding vector field is given by

$$\dot{\Phi}(0, x, y) = (1, f'(x)) = \frac{\partial}{\partial x} + f'(x)\frac{\partial}{\partial y}.$$

The corresponding ODE is written as

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = f'(x).$$

Writing the general solution curve as

$$x = x_0 + t, \quad y = y_0 - f(x_0) + f(x_0 + t),$$

we have

$$\dot{x} = 1, \quad \dot{y} = f'(x_0 + t) = f(x).$$

(ii) $\Psi(t, x, y) = (\cosh(t)x + \sinh(t)y, \sinh(t)x + \cosh(t)y)$.

SOLUTION: Again, a straightforward calculation shows that $\Psi(0, x, y) = (x, y)$. Let's verify the 1-parameter group law. We have

$$\begin{aligned}
&\Psi(s, \Psi(t, x, y)) = \Psi(s, \cosh(t)x + \sinh(t)y, \sinh(t)x + \cosh(t)y) \\
&= ((\cosh(s)\cosh(t) + \sinh(s)\sinh(t))x + (\cosh(s)\sinh(t) + \sinh(s)\cosh(t)y, \\
&\quad (\cosh(s)\sinh(t) + \sinh(s)\cosh(t))x + (\cosh(s)\cosh(t) + \sinh(s)\sinh(t)y) \\
&= (\cosh(s + t)x + \sinh(s + t)y, \sinh(s + t)x + \cosh(s + t)y) \\
&= \Psi(s + t, x, y).
\end{aligned}$$

The corresponding vector field and ODE are

$$\dot{\Psi}(0, x, y) = (y, x) = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

$$\dot{x} = y, \quad \dot{y} = x. \tag{4}$$

Writing the general solution as

$$x = \cosh(t)x_0 + \sinh(t)y_0, \quad y = \sinh(t)x_0 + \cosh(t)y_0,$$

we verify by direct calculation that (4) holds.

(c) Fix t, and let $(u, v) = \Phi_t(x, y)$ be a change of coordinates. Rewrite the corresponding vector field $\mathbf{A}(\mathbf{x}) = \dot{\Phi}(0, \mathbf{x})$ using the (u, v) coordinates. Repeat this exercise for Ψ_t and the vector field $\mathbf{B}(\mathbf{x}) = \dot{\Psi}(0, \mathbf{x})$. What does this exercise have to do with equation (3)? Discuss.

SOLUTION: Fixing t and writing

$$u = x + t, \quad v = y - f(x) + f(x + t),$$

we obtain

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + (-f'(x) + f'(x+t))\frac{\partial}{\partial v},$$
$$\frac{\partial}{\partial y} = \frac{\partial}{\partial v},$$
$$\frac{\partial}{\partial x} + f'(x)\frac{\partial}{\partial y} = \frac{\partial}{\partial u} + (f'(x) - f'(x) + f'(x+t))\frac{\partial}{\partial v} = \frac{\partial}{\partial u} + f'(u)\frac{\partial}{\partial v}$$

For the second flow we fix t and write

$$u = \cosh(t)x + \sinh(t)y, \quad v = \sinh(t)x + \cosh(t)y.$$

Hence,

$$\frac{\partial}{\partial x} = \cosh(t)\frac{\partial}{\partial u} + \sinh(t)\frac{\partial}{\partial v},$$
$$\frac{\partial}{\partial y} = \sinh(t)\frac{\partial}{\partial u} + \cosh(t)\frac{\partial}{\partial v},$$
$$y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v},$$

as was to be shown.

Recall that the transformation law for the components of a vector field is given by the pushforward of the vector field in question by the transformation that relates the two coordinate systems. Equation (3) tells us that the transformation Φ_t preserves the vector field $\dot{\Phi}_0$. Note that in both cases the form of the vector field is the same in the old and the new coordinates.

(2) (a) Calculate the flow $\Phi(t, x, y)$, respectively $\Psi(t, x, y)$, for the following vector fields:

$$\mathbf{A} = \frac{\partial}{\partial x} + 2\frac{y}{x}\frac{\partial}{\partial y}, \quad x > 0, \tag{5}$$
$$\mathbf{B} = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}. \tag{6}$$

Solution: The vector field $\mathbf{A}$ is the autonomization of the ODE

$$\frac{dy}{dx} = \frac{2y}{x}.$$

The general solution of the latter is

$$y = Kx^2.$$

Rewriting the solution as a flow, we obtain

$$\Phi(t, x, y) = (x + t, \frac{y}{x^2}(x + t)^2).$$

The vector field $\mathbf{B}$ describes the decoupled linear ODE

$$\dot{x} = x, \quad \dot{y} = 2y.$$

The corresponding flow is

$$\Psi(t, x, y) = (xe^t, ye^{2t}).$$

(b) Show that $f(x, y) = x^2/y$ is a first integral of the corresponding ODEs in two ways.

(i) Verify that

$$\mathbf{A}[x^2/y] = \mathbf{B}[x^2/y] = 0. \tag{7}$$

(ii) Verify that $f(\Phi(t,x,y))$ and $f(\Psi(t,x,y))$ are constant with respect to t.

SOLUTION: We have

$$\begin{aligned}
\mathbf{A}[x^2/y] &= \frac{\partial f(x,y)}{\partial x} + \frac{2y}{x}\frac{\partial f(x,y)}{\partial y} \\
&= \frac{2x}{y} - \frac{2y}{x}\frac{x^2}{y^2} = 0, \\
\mathbf{B}[x^2/y] &= x\frac{\partial f(x,y)}{\partial x} + 2y\frac{\partial f(x,y)}{\partial y} \\
&= \frac{2x^2}{y} - 2y\frac{x^2}{y^2} = 0.
\end{aligned}$$

Composing $f(x,y)$ with the two flows, we obtain

$$\begin{aligned}
f(\Phi(t,x,y)) &= \frac{(x+t)^2}{\frac{y}{x^2}(x+t)^2} = \frac{x^2}{y}, \\
f(\Psi(t,x,y)) &= \frac{x^2e^{2t}}{ye^{2t}} = \frac{x^2}{y}.
\end{aligned}$$

In both cases, we obtain expressions that are independent of t.

(c) Note that $\mathbf{B} = x\mathbf{A}$. Hence, there exists a reparameterization function $\tau(s,x,y)$ such that

$$\Psi(s,x,y) = \Phi(\tau(s,x,y),x,y). \tag{8}$$

Determine $\tau(s,x,y)$ using the formula given in the proof of Theorem 2.1 of the handout, and verify equation (8).

SOLUTION: Set $f(x,y) = x$. According to the proof, we first calculate

$$\begin{aligned}
\sigma(t,x,y) &= \int_0^t \frac{du}{(f\circ\Phi)(u,x,y))} \\
&= \int_0^t \frac{du}{x+u} \\
&= \ln(x+t) - \ln(x).
\end{aligned}$$

Setting $s = \sigma(t,x,y)$, and solving for t, gives

$$t = \tau(s,x,y) = xe^s - x.$$

Composing this function with the first flow gives

$$\begin{aligned}
\Phi(\tau(s,x,y),x,y) &= \left(x + xe^s - x, \frac{y}{x^2}\left(x + xe^s - x\right)^2\right) \\
&= (xe^s, ye^{2s}),
\end{aligned}$$

as was to be shown.

We could also calculate $\tau(s,x,y)$ directly by reversing the order of the vector fields. Note that $\mathbf{A} = g(x,y)\mathbf{B}$ where $g(x,y) = 1/x$. Hence,

$$\begin{aligned}
\tau(s,x,y) &= \int_0^s \frac{du}{(g\circ\Psi)(u,x,y)} \\
&= \int_0^s xe^u\,du \\
&= xe^s - x.
\end{aligned}$$

(3) (a) Let $\Phi(t,\mathbf{x})$ be an analytic (can be expanded in power series) flow on n-dimensional space and let $\mathbf{V}(\mathbf{x})$ be the corresponding vector field. Use equation (2) to show that

$$\Phi(t,\mathbf{x}) = \mathbf{x} + \mathbf{V}(\mathbf{x})\,t + (\partial\mathbf{V})(\mathbf{x})\cdot\mathbf{V}(\mathbf{x})\frac{t^2}{2} + o(t^2).$$

Solution: Taylor's theorem tells us that

$$\Phi(t,\mathbf{x}) = \Phi(0,\mathbf{x}) + \dot{\Phi}(0,\mathbf{x})\,t + \ddot{\Phi}(0,\mathbf{x})\,\frac{t^2}{2} + \ldots$$

By definition, $\dot{\Phi}(0,\mathbf{x}) = \mathbf{V}(\mathbf{x})$; this gives the expression for the coefficient of t. Taking the derivative with respect to t in (2) and using the chain rule gives

$$\ddot{\Phi}(t,\mathbf{x}) = (\partial\mathbf{V})(\Phi(t,\mathbf{x}))\cdot\dot{\Phi}(t,\mathbf{x}).$$

Setting $t=0$ gives the expression for t^2 coefficient.

(b) In the older literature, a vector field is often referred to as an *infinitesimal transformation* and as an *infinitesimal generator* of a flow. Discuss this terminology in light of the linear approximation

$$\Phi(t,\mathbf{x}) = \mathbf{x} + \mathbf{V}(\mathbf{x})\,t + o(t).$$

Answer: Note that for small times, say $t=\epsilon$, the flow is given, approximately, as

$$\Phi(\epsilon,\mathbf{x}) \approx \mathbf{x} + \epsilon\mathbf{V}(\mathbf{x}).$$

Thus for small times, the flow acts on an imaginary particle by "nudging" it in the direction specified by the vector field. Of course, this is only a linear approximation, subject to higher order corrections. These higher order corrections are also determined by the vector field, and correspond to "instantaneous course corrections" imposed by variation in the vector field. This variation is expressed by the directional derivative of the vector field with respect to itself; the corresponding expression is $(\partial\mathbf{V})\cdot\mathbf{V}$, the directional derivative of $\mathbf{V}$ with respect to itself.

(4) Consider the following systems of coordinates in 2-dimensional space:

$$u = x, \quad v = y/x \tag{9}$$

$$\hat{x} = \frac{x}{x^2+y^2}, \quad \hat{y} = \frac{y}{x^2+y^2} \tag{10}$$

(a) In each case, determine the form of the corresponding fundamental vector fields.

Solution: For the first coordinate system, we have

$$\begin{aligned} x = u, \quad y &= uv, \\ \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\begin{pmatrix} 1 & 0 \\ v & u \end{pmatrix} \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\begin{pmatrix} 1 & 0 \\ y/x & x \end{pmatrix}. \end{aligned}$$

For the second coordinate system, we have

$$
\begin{aligned}
x &= \frac{\hat{x}}{\hat{x}^2+\hat{y}^2}, \quad y = \frac{\hat{y}}{\hat{x}^2+\hat{y}^2}, \quad x^2+y^2 = 1/(\hat{x}^2+\hat{y}^2),\\
\hat{x} &= \frac{x}{x^2+y^2}, \quad \hat{y} = \frac{y}{x^2+y^2},\\
\left(\frac{\partial}{\partial \hat{x}}, \frac{\partial}{\partial \hat{y}}\right) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \begin{pmatrix} \dfrac{\hat{y}^2-\hat{x}^2}{(\hat{x}^2+\hat{y}^2)^2} & -\dfrac{2\hat{x}\hat{y}}{(\hat{x}^2+\hat{y}^2)^2} \\ -\dfrac{2\hat{x}\hat{y}}{(\hat{x}^2+\hat{y}^2)^2} & \dfrac{\hat{x}^2-\hat{y}^2}{(\hat{x}^2+\hat{y}^2)^2} \end{pmatrix}\\
&= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \begin{pmatrix} y^2-x^2 & -2xy \\ -2xy & x^2-y^2 \end{pmatrix}
\end{aligned}
$$

(b) Express each of the following vector fields using the above coordinate systems:

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \tag{11}$$

$$y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \tag{12}$$

Solution: Taking inverses of the above 2×2 matrices gives

$$
\begin{aligned}
\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) &= \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) \begin{pmatrix} 1 & 0 \\ -v/u & 1/u \end{pmatrix},\\
x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \begin{pmatrix} x \\ y \end{pmatrix}\\
&= \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) \begin{pmatrix} 1 & 0 \\ -v/u & 1/u \end{pmatrix} \begin{pmatrix} u \\ uv \end{pmatrix}\\
&= u\frac{\partial}{\partial u},\\
y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \begin{pmatrix} y \\ -x \end{pmatrix}\\
&= \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) \begin{pmatrix} 1 & 0 \\ -v/u & 1/u \end{pmatrix} \begin{pmatrix} uv \\ -u \end{pmatrix}\\
&= uv\frac{\partial}{\partial u} - (1+v^2)\frac{\partial}{\partial v}.
\end{aligned}
$$

For the second coordinate system, we have

$$
\begin{aligned}
x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} &= \left(\frac{\partial}{\partial \hat{x}}, \frac{\partial}{\partial \hat{y}}\right) \begin{pmatrix} \hat{y}^2-\hat{x}^2 & -2\hat{x}\hat{y} \\ -2\hat{x}\hat{y} & \hat{x}^2-\hat{y}^2 \end{pmatrix} \begin{pmatrix} \hat{x}/(\hat{x}^2+\hat{y}) \\ \hat{y}/(\hat{x}^2+\hat{y}^2) \end{pmatrix}\\
&= -\hat{x}\frac{\partial}{\partial \hat{x}} - \hat{y}\frac{\partial}{\partial \hat{y}},\\
y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} &= \left(\frac{\partial}{\partial \hat{x}}, \frac{\partial}{\partial \hat{y}}\right) \begin{pmatrix} \hat{y}^2-\hat{x}^2 & -2\hat{x}\hat{y} \\ -2\hat{x}\hat{y} & \hat{x}^2-\hat{y}^2 \end{pmatrix} \begin{pmatrix} \hat{y}/(\hat{x}^2+\hat{y}) \\ -\hat{x}/(\hat{x}^2+\hat{y}^2) \end{pmatrix}\\
&= \hat{y}\frac{\partial}{\partial \hat{x}} - \hat{x}\frac{\partial}{\partial \hat{y}}
\end{aligned}
$$

(5) (a) Give the definition of a group and a transformation group (use a reference of your choice).

(b) Write out the multiplication table for D_4, the group of symmetries of the square.

SOLUTION: Label the vertices of the square counterclockwise A, B, C, D, with A labelling the top, left vertex and D labelling the bottom left vertex. The 8 possible rearrangements and their labels are indicated below

AB DC	DA CB	CD BA	BC AD	BA CD	CB DA	DC AB	AD BC
1	R1	R2	R3	M1	M2	M3	M4

The group multiplication table, the column transformations are followed by the row transformations, is given below:

	1	R1	R2	R3	M1	M2	M3	M4
1	1	R1	R2	R3	M1	M2	M3	M4
R1	R1	R2	R3	1	M2	M3	M4	M1
R2	R2	R3	1	R1	M3	M4	M1	M2
R3	R3	1	R1	R2	M4	M1	M2	M3
M1	M1	M4	M3	M2	1	R3	R2	R1
M2	M2	M3	M4	M1	R1	1	R3	R2
M3	M3	M4	M1	M2	R2	R1	1	R3
M4	M4	M1	M2	M3	R3	R2	R1	1

(6) Consider the differential equation

$$\frac{dy}{dx} = -2xy^2. \tag{13}$$

(a) Autonomize this ODE and give the corresponding planar vector field. Integrate the ODE and write down the corresponding flow.

SOLUTION: The autonomized ODE and the corresponding vector field, call it $\mathbf{V}$, are:

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = -2xy^2, \tag{14}$$

$$\frac{\partial}{\partial x} - 2xy^2 \frac{\partial}{\partial y}. \tag{15}$$

The general solution of the ODE (obtained by separation of variables) is

$$y = \frac{1}{K + x^2}.$$

Rewriting this as a flow, call it $\Phi(t, x, y)$, we obtain

$$\Phi(t, x, y) = \left(x + t, \frac{1}{K + (x+t)^2} \right).$$

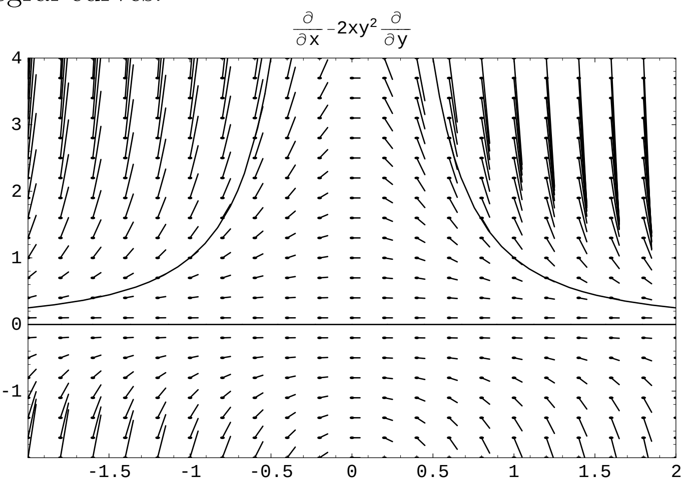
We want $\Phi(0, x, y) = (x, y)$, and this gives us

$$\frac{1}{K + x^2} = y, \quad K = \frac{1}{y} - x^2.$$

Thus, the flow is given by

$$\Phi(t, x, y) = \left(x + t, \frac{y}{y(x+t)^2 - yx^2 + 1} \right).$$

(b) Plot the vector field (Maple would be nice, otherwise graphpaper will do) and some of the integral curves.



(c) What is the domain of the flow? Determine the maximum and minimum time functions. Hint: the curves $x^2y = 1$ and $y = 0$ divide the plane into 4 regions. The domain of the flow behaves differentily in each region.

SOLUTION: There are 4 cases to consider. Case (1) corresponds to the region defined by $y \geq 0$ and $x^2y < 1$. In this case, the flow is defined for all t. Hence, the maximum and minimum time are $+\infty$ and $-\infty$, respectively.
Case (2) corresponds to the region defined by $y > 0$, $x > 0$ and $x^2y \geq 1$. For such initial conditions, the flow is defined for

$$t > \sqrt{x^2 - \frac{1}{y}} - x.$$

The minimum time is $\sqrt{x^2 - \frac{1}{y}} - x$ and the maximum time is $+\infty$.
Case (3) corresponds to the region defined by $y > 0$, $x < 0$ and $x^2y \geq 1$. For such initial conditions, the flow is defined for

$$t < -x - \sqrt{x^2 - \frac{1}{y}}.$$

The minimum time is $-\infty$. The maximum time is $-\sqrt{x^2 - \frac{1}{y}} - x$.
Case (4) corresponds to the region defined by $y < 0$. For such initial conditions, the flow is defined for

$$-\sqrt{x^2 - \frac{1}{y}} - x < t < \sqrt{x^2 - \frac{1}{y}} - x$$

The indicated bounds are the minimum and maximum time, respectively.