Assignment 2, Solutions

(1) Consider the planar vector field:

$$\boldsymbol{A} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y},$$

and the function $f(x, y) = xy$.

(a) Using directional derivatives show that $f(x, y)$ is a first integral.
SOLUTION: Calculating, we obtain

$$\boldsymbol{A}[xy] = x\partial_x(xy) - y\partial_y(xy) = xy - yx = 0.$$

(b) Use this fist integral to rectify $\boldsymbol{A}$.
SOLUTION: First, let us take $\xi = x, \eta = xy$ as coordinates. Since

$$\boldsymbol{A}[\xi] = x = \xi,$$

we have

$$\boldsymbol{A} = \xi\partial_\xi.$$

Taking coordinates

$$u = \int^\xi \frac{ds}{s} = \log(\xi) = \log(x), \quad v = \eta = xy,$$

we obtain $\boldsymbol{A} = \partial_u$.

(c) Determine the flow Φ generated by $\boldsymbol{A}$. Using an explicit calculation, verify that

$$(f \circ \Phi)(t, x, y) = f(x, y).$$

SOLUTION: The flow generated by $\boldsymbol{A}$ is obtained by integrating the ODE

$$\dot{x} = x, \quad \dot{y} = y.$$

Hence, the flow is

$$\Phi(t, x, y) = (e^t x, e^{-t} y).$$

We have

$$(f \circ \Phi)(t, x, y) = e^t x e^{-t} y = xy = f(x, y).$$

(2) Let $(r, \theta) = \boldsymbol{F}(x, y)$ be the transformation from Cartesian to polar coordinates. Let $\boldsymbol{A}$ be as in question 1, and let Φ be the flow generated by $\boldsymbol{A}$. Let $\boldsymbol{B} = \boldsymbol{F}_*\boldsymbol{A}$ and $\Psi = \boldsymbol{F}_*\Phi$ be the indicated pushforwards.

(a) Verify the principle of covariance: by direct calculation show that $\boldsymbol{B}$ generates Ψ.
SOLUTION:First, we must express $\boldsymbol{A}$ and Φ using polar coordinates. Direct calculation shows that

$$\boldsymbol{A}[r] = \frac{x^2 - y^2}{r} = r\cos(2\theta), \quad \boldsymbol{A}[\theta] = \boldsymbol{A}[\tan^{-1}(y/x)] = -\frac{2xy}{r^2} = -\sin(2\theta).$$

Hence,

$$A = r\cos(2\theta)\partial_r - 2\sin(\theta)\partial_\theta.$$

To express the flow, we write

$$\begin{aligned}\Psi(t,r,\theta) &= (\boldsymbol{F}\circ\Phi)(t, r\cos(\theta), r\sin(\theta) = \boldsymbol{F}(e^t r\cos(\theta), e^{-t} r\sin(\theta))\\ &= \left(r\sqrt{e^{2t}\cos^2(\theta)+e^{-2t}\sin^2(\theta)}, \tan^{-1}(e^{-2t}\tan(\theta))\right).\end{aligned}$$

Taking the partial derivative with respect to t and setting $t=0$ gives

$$\dot{\Psi}(0,r,\theta) = \left(r(\cos^2\theta - \sin^2\theta), \frac{2\tan(\theta)}{1+\tan^2(\theta)}\right) = (r\cos(2\theta), -\sin(2\theta)).$$

(b) Use the principle of covariance to rectify the vector field

$$\boldsymbol{B} = x\cos(2y)\frac{\partial}{\partial x} - \sin(2y)\frac{\partial}{\partial y}.$$

Solution:We recognize above the pushforward of the radial vector field via the polar coordinate transformation. We therefore introduce

$$\xi = x\cos y, \quad \eta = x\sin y$$

so that $\boldsymbol{B} = \xi\partial_\xi + \eta\partial_\eta$. We can therefore rectify $\boldsymbol{B}$ by taking coordinates

$$u = 2\eta\xi = x^2\sin(2y), \quad v = \log(\xi) = \log(x) + \log(\cos(y)).$$

In these coordinates, $\boldsymbol{B} = \partial_v$.

(3) Let $g(x,y) = x^2 - y^2$, and let $(x,y) = \boldsymbol{G}(r,\theta)$ be the transformation from polar to Cartesian coordinates. Let $\boldsymbol{A}, \boldsymbol{B}, \Phi, \Psi$ be as above.

(a) Let $h = \boldsymbol{G}^* g$. By direct calculation, verify the principle of covariance for directional derivatives by showing that

$$\boldsymbol{B}[h](r,\theta) = \boldsymbol{A}[g](x,y).$$

Solution: We have

$$\begin{aligned}h(r,\theta) &= g(x,y) = x^2 - y^2 = r^2\cos(2\theta);\\ \boldsymbol{A}[g](x,y) &= x\partial_x(x^2-y^2) - y\partial_y(x^2-y^2) = 2(x^2+y^2);\\ \boldsymbol{B}[h](r,\theta) &= r\cos(2\theta)\partial_r(r^2\cos(2\theta)) - \sin(2\theta)\partial_\theta(r^2\cos(2\theta))\\ &= 2r^2\cos^2(2\theta) + 2r^2\sin^2(2\theta)\\ &= 2r^2 = 2(x^2+y^2).\end{aligned}$$

(b) Let $u(t,x,y) = (\Phi_t^* g)(x,y)$ and $v(t,r,\theta) = (\Psi_t^* h)(r,\theta)$. Verify by explicit calculation that

$$u(t,x,y) = v(t,r,\theta).$$

Solution: We have

$$\begin{aligned} u(t,x,y) &= (g \circ \Phi_t)(x,y) = g(e^t x, e^{-t} y) = e^{2t} x^2 - e^{-2t} y^2; \\ v(t,r,\theta) &= (h \circ \Psi_t)(r,\theta) \\ &= r^2 (e^{2t} \cos^2(\theta) + e^{-2t} \sin^2(\theta)) \cos(2\phi), \end{aligned}$$

where

$$\begin{aligned} \tan(\phi) &= e^{-2t} \tan(\theta), \\ \cos^2(\phi) &= 1/(1 + e^{-4t} \tan^2(\theta)), \\ \cos(2\phi) &= (1 - e^{-4t} \tan^2(\theta))/(1 + e^{-4t} \tan^2(\theta)). \end{aligned}$$

Hence,

$$v(t,r,\theta) = r^2 (e^{2t} \cos^2(\theta) - e^{-2t} \sin^2(\theta)) = u(t,x,y)$$

(c) Finally, verify the geometric definition of the directional derivative by showing that

$$\begin{aligned} \boldsymbol{A}[g](x,y) &= \dot{u}(0,x,y) \\ \boldsymbol{B}[h](r,\theta) &= \dot{v}(0,r,\theta). \end{aligned}$$

Solution:

$$\begin{aligned} \dot{u}(0,x,y) &= 2x^2 + 2y^2; \\ \dot{v}(0,r,\theta) &= 2r^2 \cos^2(2\theta) + 2r^2 \sin^2(2\theta) = 2r^2. \end{aligned}$$

(4) Let (r,θ) be the usual polar coordinates. We showed in lecture that

$$\frac{x}{\sqrt{x^2+y^2}} \frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}} \frac{\partial}{\partial y} = \frac{\partial}{\partial r}.$$

(a) Use the above rectification to determine the flow for the vector field $\partial/\partial r$.
(b) Verify that this flow gives the general solution for the ODE

$$\dot{x} = \frac{x}{\sqrt{x^2+y^2}}, \quad \dot{y} = \frac{y}{\sqrt{x^2+y^2}}.$$

Solution: In polar coordinates, the flow is given by

$$r = r_0 + t, \quad \theta = \theta_0.$$

Converting to Cartesian coordinates, we have

$$\begin{aligned}\Phi(t, x_0, y_0) &= (x, y) = (r\cos\theta, r\sin\theta)\\ &= ((r_0+t)\cos(\theta_0), (r_0+t)\sin(\theta_0))\\ &= \left(x_0 + \frac{tx_0}{\sqrt{x_0^2+y_0^2}}, y_0 + \frac{ty_0}{\sqrt{x_0^2+y_0^2}}\right)\end{aligned}$$

We then have

$$\begin{aligned}&\sqrt{x^2+y^2} = t + \sqrt{x_0^2+y_0^2};\\ &\frac{x}{\sqrt{x^2+y^2}} = \frac{x_0(t+\sqrt{x_0^2+y_0^2})}{\sqrt{x_0^2+y_0^2}}\frac{1}{t+\sqrt{x_0^2+y_0^2}} = \frac{x_0}{\sqrt{x_0^2+y_0^2}}\end{aligned}$$

It now follows that

$$\dot{x} = \frac{x_0}{\sqrt{x_0^2+y_0^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

Similarly,

$$\dot{y} = \frac{y_0}{\sqrt{x_0^2+y_0^2}} = \frac{y}{\sqrt{x^2+y^2}}$$

(5) Prove Proposition 5.4 of the class notes (coherence of pullback and push-forward).
SOLUTION:The coherence of the pullback follows directly from the definitions. Let $\boldsymbol{F}, \boldsymbol{G}, h$ be as in the Proposition. Set $g = \boldsymbol{G}^* h$ and $f = \boldsymbol{F}^* g = \boldsymbol{F}^*(\boldsymbol{G}^* h)$. In other words,

$$\begin{aligned}g(\boldsymbol{y}) &= h(\boldsymbol{G}(y)), \quad y \in V,\\ f(\boldsymbol{x}) &= g(\boldsymbol{F}(\boldsymbol{x})) = h(\boldsymbol{G}(\boldsymbol{F}(\boldsymbol{x}))) = (h \circ \boldsymbol{G} \circ \boldsymbol{F})(x), \quad \boldsymbol{x} \in U,\\ &= ((\boldsymbol{G} \circ \boldsymbol{F})^* h)(\boldsymbol{x}).\end{aligned}$$

The coherence of the push-forward follows from the chain rule. For convenience, set $\boldsymbol{B} = \boldsymbol{F}_* \boldsymbol{A}$ and $\boldsymbol{C} = \boldsymbol{G}_* \boldsymbol{B} = \boldsymbol{G}_*(\boldsymbol{F}_* \boldsymbol{A})$. By definition of push-forward,

$$\begin{aligned}\boldsymbol{B} \circ \boldsymbol{F} &= \mathfrak{J}\boldsymbol{F} \cdot \boldsymbol{A},\\ \boldsymbol{C} \circ \boldsymbol{G} &= \mathfrak{J}\boldsymbol{G} \cdot \boldsymbol{B}.\end{aligned}$$

It follows that

$$\begin{aligned}\boldsymbol{C} \circ \boldsymbol{G} \circ F &= (\mathfrak{J}\boldsymbol{G} \circ \boldsymbol{F}) \cdot (\boldsymbol{B} \circ \boldsymbol{F})\\ &= (\mathfrak{J}\boldsymbol{G} \circ F) \cdot \mathfrak{J}\boldsymbol{F} \cdot \boldsymbol{A}.\end{aligned}$$

The chain rule tells us that

$$\mathcal{J}(\boldsymbol{G} \circ \boldsymbol{F}) = (\mathcal{J}\,\boldsymbol{G} \circ F) \cdot \mathcal{J}\,\boldsymbol{F}.$$

Hence,

$$\boldsymbol{C} \circ \boldsymbol{G} \circ F = \mathcal{J}(\boldsymbol{G} \circ \boldsymbol{F}) \cdot \boldsymbol{A}.$$

Therefore,

$$\boldsymbol{C} = (\boldsymbol{G} \circ \boldsymbol{F})_* \boldsymbol{A},$$

as was to be shown.

(6) In lecture we showed that the vector field

$$\boldsymbol{A} = \frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y}$$

generates the flow

$$\Phi(t, x, y) = (x + t, (x + y + 1)e^t - x - t - 1).$$

(a) Use the flow to determine a first integral of $\boldsymbol{A}$.
Solution: We solve the equation $\Phi^1(\tau, x, y) = 0$ and substitute the solution $\tau = f(x, y)$ into Φ^2 to obtain the first integral $I = \Phi^2(\tau, x, y)$. We have

$$\Phi^1(\tau, x, y) = x + \tau = 0,$$

whence $\tau = -x$. Hence, a first integral is given by

$$\eta = (x + y + 1)e^{-x} - 1.$$

Let us verify our answer:

$$\begin{aligned}\boldsymbol{A}[\eta] &= \partial_x((x + y + 1)e^{-x} - 1) + (x + y)\partial_y((x + y + 1)e^{-x} - 1)\\ &= e^{-x} - e^{-x}(x + y + 1) + (x + y)e^{-x} = 0.\end{aligned}$$

(b) Rectify $\boldsymbol{A}$.
Solution: Let use coordinates

$$\xi = x, \quad \eta = (x + y + 1)e^{-x} - 1.$$

Since $\boldsymbol{A}[\xi] = 1$ and $\boldsymbol{A}[\eta] = 0$, we have $\boldsymbol{A} = \partial_\xi$.