Assignment 3, Solutions

(1) Consider the following planar vector fields:

$$\boldsymbol{A} = x\partial_x - y\partial_y, \qquad \boldsymbol{B} = -y\partial_x + x\partial_y.$$

(a) Reexpress $\boldsymbol{A}, \boldsymbol{B}$ in polar coordinates.
SOLUTION:

$$\begin{aligned}
x\partial_x - y\partial_y &= (xr_x - yr_y)\,\partial_r + (x\theta_x - y\theta_y)\,\partial_\theta \\
&= \frac{x^2 - y^2}{\sqrt{x^2+y^2}}\partial_r - \frac{2xy}{x^2+y^2}\partial_\theta \\
&= r\cos(2\theta)\partial_r - \sin(2\theta)\partial_\theta; \\
-y\partial_x + x\partial_y &= \partial_\theta.
\end{aligned}$$

(b) Calculate the Lie bracket using Cartesian and polar coordinates, and verify the principle of covariance by showing that the two calculations agree.
SOLUTION:

$$\begin{aligned}
[r\cos(2\theta)\partial_r - \sin(2\theta)\partial_\theta, \partial_\theta] &= -\left[\partial_\theta, r\cos(2\theta)\partial_r - \sin(2\theta)\partial_\theta\right] \\
&= 2r\sin(2\theta)\partial_r + 2\cos(2\theta)\partial_\theta. \\
[x\partial_x - y\partial_y, -y\partial_x + x\partial_y] &= x\partial_y + y\partial_x + y\partial_x + x\partial_y \\
&= 2x\partial_y + 2y\partial_x \\
&= (yr_x + xr_y)\,\partial_r + (y\theta_x + x\theta_y)\,\partial_\theta \\
&= \frac{2xy}{\sqrt{x^2+y^2}}\partial_r + \frac{x^2-y^2}{x^2+y^2}\partial_\theta \\
&= 2r\sin(2\theta)\partial_r + 2\cos(2\theta)\partial_\theta.
\end{aligned}$$

(2) Let $\boldsymbol{A}, \boldsymbol{B}$ be the vector fields of question 1 and let Φ_t, Ψ_t be the corresponding flows.
(a) Calculate the time dependent vector fields

$$\boldsymbol{C}_t = \Phi_{t*}\boldsymbol{B}, \qquad \boldsymbol{E}_t = \Psi_{t*}\boldsymbol{A}$$

SOLUTION: Recall that

$$\begin{aligned}
\Phi(t,x,y) &= (e^t x, e^{-t} y) \\
\Psi(t,x,y) &= (\cos(t)x - \sin(t)y, \sin(t)x + \cos(t)y)
\end{aligned}$$

Hence,

$$\begin{aligned}
\boldsymbol{C}_t &= \Phi_{t*}\boldsymbol{B} = (\partial\!\!\!/\,\Phi_t \cdot \boldsymbol{B}) \circ \Phi_{-t} \\
&= \left(\begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} -y \\ x \end{pmatrix} \right) \circ \Phi_{-t} \\
&= -e^{2t} y \partial_x + e^{-2t} x \partial_y . \\
\boldsymbol{E}_t &= \Psi_{t*}\boldsymbol{A} = (\partial\!\!\!/\,\Psi_t \cdot \boldsymbol{A}) \circ \Psi_{-t} \\
&= \left(\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} \right) \circ \Psi_{-t} \\
&= (\cos(2t)x + \sin(2t)y)\partial_x + (\sin(2t)x - \cos(2t)y)\partial_y
\end{aligned}$$

(b) Verify that

$$\begin{aligned}
\dot{\boldsymbol{C}}_t &= [\boldsymbol{C}_t, \boldsymbol{A}] \\
\dot{\boldsymbol{E}}_t &= [\boldsymbol{E}_t, \boldsymbol{B}].
\end{aligned}$$

SOLUTION: We have

$$\begin{aligned}
[\boldsymbol{C}_t, \boldsymbol{A}] &= [-e^{2t} y \partial_x + e^{-2t} x \partial_y, x\partial_x - y\partial_y] \\
&= -2e^{2t} y \partial_x - 2e^{-2t} x \partial_y . \\
[\boldsymbol{E}_t, \boldsymbol{B}] &= [(\cos(2t)x + \sin(2t)y)\partial_x + (\sin(2t)x - \cos(2t)y)\partial_y, -y\partial_x + x\partial_y] \\
&= (\cos(2t)x + \sin(2t)y)\partial_y + (-\sin(2t)x + \cos(2t)y)\partial_x \\
&\quad + (y\cos(2t) - \sin(2t)x)\partial_x + (\sin(2t)y + \cos(2t)x)\partial_y \\
&= 2(-\sin(2t)x + \cos(2t)y)\partial_x + 2(\cos(2t)x + \sin(2t)y)\partial_y
\end{aligned}$$

(c) Finally, verify that

$$\begin{aligned}
\dot{\boldsymbol{C}}_0 &= [\boldsymbol{B}, \boldsymbol{A}] \\
\dot{\boldsymbol{E}}_0 &= [\boldsymbol{A}, \boldsymbol{B}].
\end{aligned}$$

SOLUTION: Set $t = 0$ in the above formulas. By inspection $\boldsymbol{C}_0 = \boldsymbol{B}$ and $\boldsymbol{E}_0 = \boldsymbol{A}$.

(3) (a) Find the most general infinitesimal symmetry of ∂_x. In other words, describe all planar vector fields $\boldsymbol{A}$ such that

$$[\boldsymbol{A}, \partial_x] = 0.$$

SOLUTION: Write $\boldsymbol{A} = \xi\partial_x + \eta\partial_y$. Note that

$$[\boldsymbol{A}, \partial_x] = -[\partial_x, \boldsymbol{A}] = -\xi_x \partial_x - \eta_x \partial_y .$$

Hence $\boldsymbol{A}$ is a symmetry if and only if ξ and η are functions of y only, say $\xi = a(y),\ \eta = b(y)$.

(b) Find the most general symmetry of ∂_x. In other words, find the most general transformation $(u, v) = \boldsymbol{F}(x, y)$ such that $\boldsymbol{F}_* \partial_x = \partial_x$, or what is equivalent such that $\partial_x = \partial_u$.
SOLUTION: We are seeking functions $u = F^1(x, y)$ and $v = F^2(x, y)$ such that

$$\partial_x(u) = 1, \quad \partial_x(v) = 0.$$

Hence v is a function of y only, say $v = f(y)$, such that $f'(y) \neq 0$. As for the first condition, it implies that

$$u = x + g(y)$$

where $g(y)$ is an arbitrary (but sufficiently differentiable) function of 1 variable.

(c) Find the most general infinitesimal conformal symmetry of ∂_x. In other words, describe all planar vector fields $\boldsymbol{A}$ such that

$$[\boldsymbol{A}, \partial_x] = f(x, y)\partial_x.$$

SOLUTION: Write $\boldsymbol{A}$ as above. We want $\partial_x \eta = 0$. Therefore, $\boldsymbol{A}$ is a conformal symmetry if and only if it has the form

$$\boldsymbol{A} = A(x, y)\partial_x + b(y)\partial_y,$$

where A is an arbitrary C^1 function of two variables related to $f(x, y)$ according to

$$A(x, y) = -\int^x f(u, y)du,$$

and where $b(y)$ is an arbitrary C^1 function of one variable.

(d) Find the most general form of an ODE $dy/dx = \omega(x, y)$ for which ∂_x is an infinitesimal symmetry.
SOLUTION: We ask that $[\partial_x, \partial_x + \omega(x, y)\partial_y] = \omega_x \partial_y$ be proportional to $\partial_x + \omega\partial_y$. This is true if and only if ω is a function of y only. Thus ∂_x is a symmetry of ODE having the form

$$\frac{dy}{dx} = f(y),$$

where $f(y)$ is a sufficiently differentiable function of 1 variable.

(4) (a) What is the most general vector field $\boldsymbol{A}$ for which $-y\partial_x + x\partial_y$ is an infinitesimal symmetry? In other words, describe all planar vector fields $\boldsymbol{A}$ such that

$$[-y\partial_x + x\partial_y, \boldsymbol{A}] = 0.$$

Hint: switch to polar coordindates and utilize the principle of covariance.
SOLUTION: We use polar coordinates and write $\boldsymbol{A} = A(r, \theta)\partial_r + B(r, \theta)\partial_\theta$. Since $-y\partial_x + x\partial_y = \partial_\theta$, we have, by the principle of covariance,

$$[\partial_\theta, \boldsymbol{A}] = (\mathrm{D}_2\, A)(r, \theta)\partial_r + (\mathrm{D}_2\, B)(r, \theta)\partial_\theta.$$

Hence, the above bracket vanishes if and only if

$$\begin{aligned}\boldsymbol{A} &= a(r)\partial_r + b(r)\partial_\theta \\ &= a(\sqrt{x^2+y^2})\left(\frac{x}{\sqrt{x^2+y^2}}\partial_x + \frac{y}{\sqrt{x^2+y^2}}\partial_y\right) + \\ &\quad + b(\sqrt{x^2+y^2})\,(-y\partial_x + x\partial_y) \\ &= (x\tilde{a}(x^2+y^2) - y\tilde{b}(x^2+y^2)\partial_x + (y\tilde{a}(x^2+y^2) + x\tilde{b}(x^2+y^2))\partial_y,\end{aligned}$$

where $\tilde{a}, \tilde{b}$ are arbitrary C^1 functions of one variable and where

$$a(u) = u\,\tilde{a}(u^2), \quad b(u) = \tilde{b}(u^2).$$

(b) What is the most general vector field $\boldsymbol{A}$ for which $-y\partial_x + x\partial_y$ is an infinitesimal conformal symmetry? In other words, describe all planar vector fields $\boldsymbol{A}$ such that $[-y\partial_x + x\partial_y, \boldsymbol{A}]$ is proportional to $\boldsymbol{A}$.

SOLUTION: Proceeding as above, we see that A, B must be functions of two variables that satisfy

$$(\mathrm{D}_2 A)B = (\mathrm{D}_2 B)A.$$

Equivalently,

$$\mathrm{D}_2 \ln(A/B) = 0,$$

which means that

$$A(r,\theta) = B(r,\theta)g(r),$$

where g is an arbitrary function of one variable. Therefore, $-y\partial_x + x\partial_y$ is a conformal symmetry of $\boldsymbol{A}$ if and only if

$$\begin{aligned}\boldsymbol{A} &= B(r,\theta)\,(g(r)\partial_r + \partial_\theta) \\ &= \tilde{B}(x,y)g(\sqrt{x^2+y^2})\left(\frac{x}{\sqrt{x^2+y^2}}\partial_x + \frac{y}{\sqrt{x^2+y^2}}\partial_y\right) + \tilde{b}(x,y)\,(-y\partial_x + x\partial_y) \\ &= \tilde{B}(x,y)\left((x\tilde{g}(x^2+y^2) - y)\partial_x + (-y\tilde{g}(x^2+y^2) + x)\partial_y\right),\end{aligned}$$

where $\tilde{B}(x,y) = B(r,\theta)$ is an arbitrary function of 2 variables, where $\tilde{g}$ is an arbitrary function of one variable, and where $g(u) = u\,\tilde{g}(u^2)$.

Here's is an alternative solution. We showed in class that a vector field $\boldsymbol{B}$ is a conformal symmetry of $\boldsymbol{A}$ if and only if there exists a function f such that $[\boldsymbol{B}, f\boldsymbol{A}] = 0$. This means that it suffices to take the general solution of the preceding question, and multiply it by an arbitrary function of two variables. Thus, the desired $\boldsymbol{A}$ has the form

$$\boldsymbol{A} = f(x,y)\left((x\tilde{a}(x^2+y^2) - y\tilde{b}(x^2+y^2)\partial_x + (y\tilde{a}(x^2+y^2) + x\tilde{b}(x^2+y^2))\partial_y\right),$$

where f is an arbitrary function of 2 variables, and $\tilde{a}, \tilde{b}$ are arbitrary functions of 1 variable. However, this answer is not entirely satisfactory, because we can absorb either $\tilde{a}$ or $\tilde{b}$ into f as follows:

$$\boldsymbol{A} = f(x,y)\tilde{b}(x^2+y^2)\left(\left(x\frac{\tilde{a}(x^2+y^2)}{\tilde{b}(x^2+y^2)} - y\right)\partial_x + \left(y\frac{\tilde{a}(x^2+y^2)}{\tilde{b}(x^2+y^2)} + x\right)\partial_y\right).$$

In other words, by setting

$$\tilde{B}(x,y) = f(x,y)\tilde{b}(x^2+y^2), \qquad \tilde{g}(u) = \frac{\tilde{a}(u)}{\tilde{b}(u)}$$

we recover the preceding answer.

(5) Prove that the Lie bracket obeys the Jacobi identity.
SOLUTION: Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be vector fields. We use the commutator relation to prove the Jacobi identity

$$[\boldsymbol{A},[\boldsymbol{B},\boldsymbol{C}]] + [\boldsymbol{B},[\boldsymbol{C},\boldsymbol{A}]] + [\boldsymbol{C},[\boldsymbol{A},\boldsymbol{B}]] = 0.$$

Let f be a twice differentiable function. We then have

$$\begin{aligned} [\boldsymbol{B},\boldsymbol{C}][f] &= \boldsymbol{B}[\boldsymbol{C}[f]] - \boldsymbol{C}[\boldsymbol{B}[f]] \\ [\boldsymbol{A},[\boldsymbol{B},\boldsymbol{C}]][f] &= \boldsymbol{A}[[\boldsymbol{B},\boldsymbol{C}][f]] - [\boldsymbol{B},\boldsymbol{C}][\boldsymbol{A}[f]] \qquad (1) \\ &= \boldsymbol{A}[\boldsymbol{B}[\boldsymbol{C}[f]]] - \boldsymbol{A}[\boldsymbol{C}[\boldsymbol{B}[f]]] - \boldsymbol{B}[\boldsymbol{C}[\boldsymbol{A}[f]]] + \boldsymbol{C}[\boldsymbol{B}[\boldsymbol{A}[f]]] \qquad (2) \end{aligned}$$

Similarly,

$$\begin{aligned} [\boldsymbol{B},[\boldsymbol{C},\boldsymbol{A}]][f] &= \boldsymbol{B}[\boldsymbol{C}[\boldsymbol{A}[f]]] - \boldsymbol{B}[\boldsymbol{A}[\boldsymbol{C}[f]]] - \boldsymbol{C}[\boldsymbol{A}[\boldsymbol{B}[f]]] + \boldsymbol{A}[\boldsymbol{C}[\boldsymbol{B}[f]]] \qquad (3) \\ [\boldsymbol{C},[\boldsymbol{A},\boldsymbol{B}]][f] &= \boldsymbol{C}[\boldsymbol{A}[\boldsymbol{B}[f]]] - \boldsymbol{C}[\boldsymbol{B}[\boldsymbol{A}[f]]] - \boldsymbol{A}[\boldsymbol{B}[\boldsymbol{C}[f]]] + \boldsymbol{B}[\boldsymbol{A}[\boldsymbol{C}[f]]] \qquad (4) \end{aligned}$$

Adding (1) (3) (4) together, gives the Jacobi identity.

(6) Prove the following Proposition. Let $\boldsymbol{F}: U \to V$ be a diffeomorphism and $\boldsymbol{G}: V \to U$ the inverse transformation. For $\boldsymbol{A}: U \to \mathbb{R}^n$, a vector field, and $f: U \to \mathbb{R}$, a function

$$\boldsymbol{F}_*(g\boldsymbol{B}) = \boldsymbol{G}^*(g)\boldsymbol{F}_*(\boldsymbol{B}). \qquad (5)$$

(7) A Bernoulli equation is a first-order ODE having the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \quad n = 1, 2, \ldots$$

(a) Consult a standard ODE text and describe the usual "cookbook" procedure for integrating a Bernoulli equation.
SOLUTION: One rewrites the equation as

$$\frac{d}{dx}(y^{1-n}) + (1-n)p(x)y^{1-n} = (1-n)q(x)$$

The above is an inhomogeneous linear equation. The integrating factor is $\phi(x)$ where

$$\phi(x) = \exp\left((1-n)\int p(x)\,dx\right), \tag{6}$$

or equivalently,

$$\frac{\phi'(x)}{\phi(x)} = (1-n)p(x). \tag{7}$$

The differential equation can now be written as

$$\frac{d}{dx}\left(\phi(x)y^{1-n}\right) = (1-n)\phi(x)q(x).$$

The solution can now be given as

$$y = \left(\frac{(1-n)\int \phi(x)q(x)\,dx + C}{\phi(x)}\right)^{\frac{1}{1-n}}.$$

(b) Show that a Bernoulli equation admits a symmetry of the form

$$f(x)y^n\partial_y.$$

Recover the standard integration method by using the symmetry.

Solution: We must show that the above vector field is a conformal symmetry of the vector field $\partial_x + (-p(x)y + q(x)y^n)\partial_y$. Calculating the Lie bracket gives

$$[f(x)y^n\partial_y, \partial_x + (-p(x)y + q(x)y^n)\partial_y] = ((n-1)f(x)p(x) - f'(x))\,y^n\partial_y$$

Therefore, if

$$\frac{f'(x)}{f(x)} = (n-1)p(x),$$

or what is equivalent, if $f(x) = 1/\phi(x)$ where $\phi(x)$ is given by (6), then the given vector field is a conformal symmetry of the Bernoulli equation. Writing the ODE in differential form

$$dy + (p(x)y - q(x)y^n)dx = 0$$

and contracting the left-hand side with the symmetry vector field gives $\phi(x)/y^n$ as the integrating factor. Multiplying through and using (7) gives the ODE in exact form:

$$\frac{dy}{f(x)y^n} + \left(\frac{f'(x)y^{1-n}}{(n-1)f(x)^2} - \frac{q(x)}{f(x)}\right)dx = 0.$$

Integrating the above gives

$$\frac{y^{1-n}}{(1-n)f(x)} - \int^x \frac{q(u)}{f(u)}\,du = C$$

By inspection, the above is equivalent to the solution shown in part (a).

(8) (a) Describe the most general first-order ODE that admits $-y\partial_x + x\partial_y$ as a symmetry. Describe the integration procedure for such equations. Hint: see question 4.

SOLUTION: The most general vector field that commutes with the above vector field is

$$\begin{aligned}\partial_\theta + \tilde{f}(r)\partial_r &= -y\partial_x + x\partial_y + f(x^2+y^2)\,(x\partial_x + y\partial_y)\,,\\ &= (f(x^2+y^2)x - y)\partial_x + (x + f(x^2+y^2)y)\partial_y\\ &= (f(x^2+y^2)x - y)\left(\partial_x + \frac{x + f(x^2+y^2)y}{f(x^2+y^2)x - y}\partial_y\right)\end{aligned}$$

where (r,θ) are the usual polar coordinates, and where $\tilde{f}(r) = rf(r^2)$. Therefore, the most general ODE preserved by $-y\partial_x + x\partial_y = \partial_\theta$ has the form

$$\frac{dy}{dx} = \frac{x + f(x^2+y^2)y}{f(x^2+y^2)x - y}.$$

In differential form the above ODE reads

$$\boldsymbol{\alpha} = (f(x^2+y^2)x - y)dy - (x + f(x^2+y^2)y)dx = 0.$$

Contracting the LHS with the symmetry ODE and taking the reciprocal gives the integrating factor, namely

$$\frac{1}{\boldsymbol{\alpha}(\boldsymbol{A})} = \frac{1}{(x^2+y^2)f(x^2+y^2)}.$$

Therefore, the exact form of the equation is

$$-\left(\frac{y}{x^2+y^2} + \frac{x}{(x^2+y^2)f(x^2+y^2)}\right)dx + \left(\frac{x}{x^2+y^2} - \frac{y}{(x^2+y^2)f(x^2+y^2)}\right)dy$$

Integrating the above yields

$$\tan^{-1}(y/x) - \frac{1}{2}\int^{x^2+y^2}\frac{du}{uf(u)} = C$$

(b) Use this method to integrate

$$\frac{dy}{dx} = \frac{x^3 + xy^2 + y}{x - y^3 - x^2y} = \frac{x(x^2+y^2) + y}{x - y(x^2+y^2)}.$$

SOLUTION: To put the above equation in the form of part (a) we take $f(u) = 1/u$. Applying the above method gives the (implicit) solution

$$\tan^{-1}(y/x) - \frac{1}{2}(x^2+y^2) = C.$$

Using polar coordinates, we see that the solution curves (see Figure 1) are spirals, described by the equation

$$r = \pm\sqrt{2\theta - 2C}, \quad \theta > C.$$

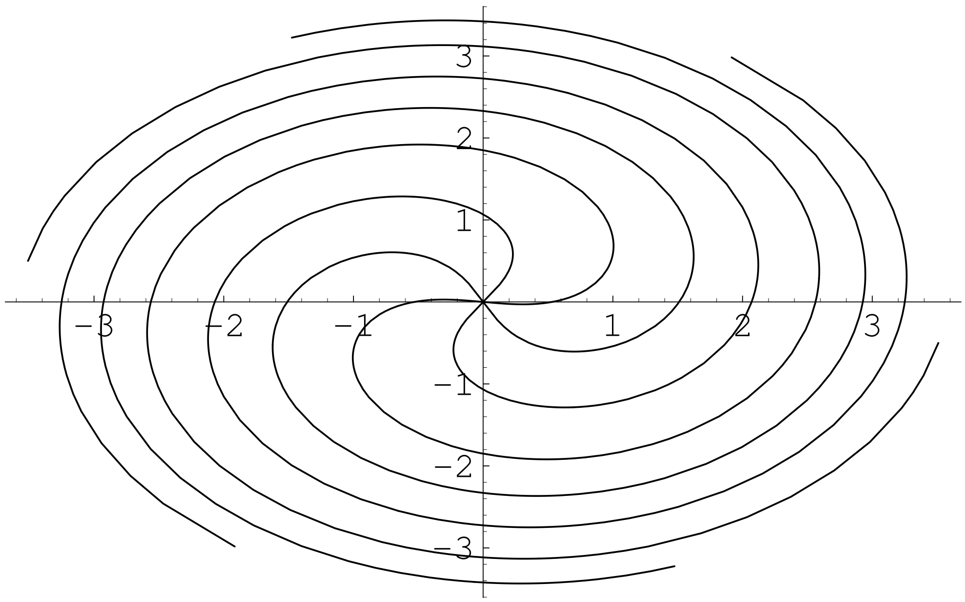

FIGURE 1. $r = \pm\sqrt{2\theta - 2C}$

(9) (a) Let $\boldsymbol{A} = g(x)(x\partial_x + y\partial_y)$. Show that

$$\frac{dy}{dx} = y/x + \frac{f(y/x)}{g(x)}$$

is the most general ODE that admits $\boldsymbol{A}$ as an infinitesimal symmetry. Hint: review the derivation of symmetries for homogeneous first-order ODEs.

SOLUTION: Recall that

$$[x\partial_x + y\partial_y, x\partial_y] = 0.$$

Hence, $\boldsymbol{A}$ commutes with $x\partial_y$ also. Since, y/x is annihilated by $\boldsymbol{A}$, the most general vector field commuting with $\boldsymbol{A}$ has the form

$$\begin{aligned}(x\partial_x + y\partial_y)/g(x) + \tilde{f}(y/x)\, x\partial_y &= (x/g(x))\partial_x + \Big(\tilde{f}(y/x)\, y + (y/g(x))\Big)\partial_y \\ &= (x/g(x))\left\{\partial_x + (y/x + f(y/x)/g(x))\,\partial_y\right\},\end{aligned}$$

where $f(u) = \tilde{f}(u)u$. Therefore, the most general ODE admitting $\boldsymbol{A}$ as a symmetry has the form indicated above.

(b) Describe the corresponding integration method.

SOLUTION: In differential form the above ODE reads

$$\boldsymbol{\alpha} = dy - (y/x + f\,(y/x)\,g(x))\,dx = 0.$$

Contracting $\boldsymbol{\alpha}$ with the symmetry generator $\boldsymbol{A}$ and taking the reciprocal gives the integrating factor, namely

$$\frac{1}{\boldsymbol{\alpha}(\boldsymbol{A})} = \frac{g(x)}{y - (y + f(y/x)g(x)x)} = -\frac{1}{xf(y/x)}.$$

Therefore, the exact form of the equation is

$$\left(\frac{g(x)}{x} + \frac{y}{x^2 f(y/x)}\right) dx - \frac{dy}{xf(y/x)}$$

Integrating the above yields

$$\int \frac{g(x)}{x}\,dx - \int^{y/x} \frac{du}{f(u)} = C$$

(c) Use the method to integrate the ODE

$$\frac{dy}{dx} = y/x + e^{y/x}\ln(x).$$

SOLUTION: We apply the above method with $f(u) = e^u$ and $g(x) = \ln(x)$. Solving for y, we obtain

$$y = -x\log\left(C - \frac{1}{2}\log(x)^2\right).$$