Assignment 4, Solutions

(1) (a) Let $U, V, W \subset \mathbb{R}^2$ be 2-dimensional domains and $\boldsymbol{F} : U \to V$ and $\boldsymbol{G} : V \to W$ diffeomorphisms. Prove that

$$(\boldsymbol{G} \circ \boldsymbol{F})^{(1)} = \boldsymbol{G}^{(1)} \circ \boldsymbol{F}^{(1)}$$

Hint: note that for $(u, v) = \boldsymbol{F}(x, y)$ the transformation of the derivative

$$v_1 = \frac{v_x + v_y y_1}{u_x + u_y y_1}$$

is a fractional linear transformation.

Solution: Let us write $(\xi, \eta, \eta_1) = \boldsymbol{G}(u, v, v_1)$ where $(\xi, \eta) = \boldsymbol{G}(u, v)$ and where

$$\eta_1 = \frac{\eta_u + \eta_v v_1}{\xi_u + \xi_v v_1},$$

and where

$$\eta_u = \frac{\partial \eta}{\partial u}, \quad \eta_v = \frac{\partial \eta}{\partial v}, \quad \text{etc.}$$

We also have $(\xi, \eta) = (\boldsymbol{G} \circ \boldsymbol{F})(x, y)$ and

$$\mathfrak{J}(\boldsymbol{G} \circ \boldsymbol{F})(x, y) = \begin{pmatrix} \partial\xi/\partial x & \partial\xi/\partial y \\ \partial\eta/\partial x & \partial\eta/\partial y \end{pmatrix}.$$

Therefore, we want to show that

$$\eta_1 = \frac{\eta_x + \eta_y y_1}{\xi_x + \xi_y y_1} \tag{1}$$

As for fractional linear transformations, recall that if we compose the transformations

$$w = \frac{c + dz}{a + bz}, \qquad v = \frac{\gamma + \delta w}{\alpha + \beta w},$$

we obtain the transformation

$$\begin{aligned} v &= \frac{\gamma(a + bz) + \delta(c + dz)}{\alpha(a + bz) + \beta(c + dz)} \\ &= \frac{p + qz}{r + sz}, \end{aligned}$$

where

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The chain rule gives us

$$\mathfrak{J}(\boldsymbol{G} \circ \boldsymbol{F}) = (\mathfrak{J}\,\boldsymbol{G} \circ \boldsymbol{F}) \cdot \mathfrak{J}\,\boldsymbol{F}$$

or equivalently

$$\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} = \begin{pmatrix} \xi_u & \xi_v \\ \eta_u & \eta_v \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

because

$$(\mathcal{J}\,\boldsymbol{G} \circ \boldsymbol{F})(x,y) = \begin{pmatrix} \frac{\partial \xi}{\partial u} & \frac{\partial \xi}{\partial v} \\ \frac{\partial \eta}{\partial u} & \frac{\partial \eta}{\partial v} \end{pmatrix}, \quad \mathcal{J}\,\boldsymbol{F}(x,y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

Equation (1) follows immediately.

(b) Let $\boldsymbol{A} = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ be a C^2 vector-field defined on a 2-dimensional domain with coordinates (x,y). Let Φ_t be the C^2 flow generated by $\boldsymbol{A}$. Prove that the 1-parameter family of prolonged transformations $\Phi_t^{(1)}$ defines a flow.

Solution: Let us write $(u,v,v_1) = \Phi_t^{(1)}(x,y,y_1)$ where $(u,v) = \Phi_t(x,y)$ and where

$$v_1 = \nu(t,x,y,y_1) = \frac{v_x + y_1 v_y}{u_x + y_1 u_y},$$

as per the prolongation formula. We wish to prove two claims: First, that

$$\nu(0,x,y,y_1) = y_1$$

and second, that

$$\Phi_{t+s}^{(1)} = \Phi_t^{(1)} \circ \Phi_s^{(1)},$$

or equivalently that

$$\nu(t+s,x,y,y_1) = \nu(t,u_s,v_s,v_{1s}),$$

where as usual we use the notation $f_t(x,y) = f(t,x,y)$. Regarding the first claim, observe that

$$u_0 = x, \quad v_0 = y.$$

We immediately have $v_1|_{t=0} = y_1$. The second claim follows from the fact that $\Phi_{t+s} = \Phi_t \circ \Phi_s$ and from part (a).

(c) Prove that the prolonged vector field $\boldsymbol{A}^{(1)}$ generates the prolonged flow $\Phi_t^{(1)}$. Hint: first, write $(u,v) = \Phi_t(x,y)$ and prove that

$$\begin{pmatrix} \dot{u}_x & \dot{u}_y \\ \dot{v}_x & \dot{v}_y \end{pmatrix}\Big|_{t=0} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}.$$

Solution: Since we have already that $\Phi_t^{(1)}$ is a flow, by the uniquenes of flows, the assertion in question reduces the claim that

$$\dot{v}_1|_{t=0} = (\mathrm{D}_1\nu)(0,x,y,y_1) = \eta_x + y_1(\eta_y - \xi_x) - y_1^2\xi_y.$$

Note the above RHS is the ∂_{y_1} component of $\boldsymbol{A}^{(1)}$. Recall that

$$\dot{\Phi}_t = \boldsymbol{A} \circ \Phi_t.$$

Since the partial derivative with respect to time and the partial derivatives with respect to position commute, we have by the multi-variable chain rule,

$$\begin{pmatrix} \dot{u}_x & \dot{u}_y \\ \dot{v}_x & \dot{v}_y \end{pmatrix} = \mathcal{J}\,\dot{\Phi}_t = (\mathcal{J}\boldsymbol{A} \circ \Phi_t) \cdot \mathcal{J}\,\Phi_t$$

Since Φ_0 is the identity transformation, and $\mathcal{J}\,\Phi_0$ the identity matrix, we have

$$\begin{pmatrix} \dot{u}_x & \dot{u}_y \\ \dot{v}_x & \dot{v}_y \end{pmatrix}\Big|_{t=0} = \mathcal{J}\boldsymbol{A} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}$$

Observe that

$$\dot{v}_1 = (\mathrm{D}_1\nu)(t,x,y,y_1) = \frac{\dot{v}_x + y_1\dot{v}_y}{u_x + y_1 u_y} - \frac{(v_x + y_1 v_y)(\dot{u}_x + y_1\dot{u}_y)}{(u_x + y_1 u_y)^2},$$

Since $\mathcal{J}\,\Phi_0$ is the identity transformation, we have

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}\Big|_{t=0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore,

$$\dot{v}_1|_{t=0} = (\mathrm{D}_1\nu)(0,x,y,y_1) = \eta_x + y_1\eta_y - y_1(\xi_x + y_1\xi_y),$$

as was to be shown.

(2) Let $\boldsymbol{A} = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ be a vector field on a 2-dimensional domain with coordinates (x,y). Let

$$\boldsymbol{A}^{(1)} = \xi(x,y)\partial_x + \eta(x,y)\partial_y + \nu(x,y,y_1)\partial_{y_1}$$

where

$$\nu(x,y,y_1) = (\partial_x + y_1\partial_y)[\eta(x,y) - y_1\xi(x,y)]$$

be the prolongation of $\boldsymbol{A}$, a vector field in the 3-dimensional domain with coordinates (x,y,y_1). Let S_ω be the surface in this 3-dimensional domain defined by the equation $y_1 = \omega(x,y)$. Prove that $\boldsymbol{A}$ is an infinitesimal symmetry of a 1st-order ODE

$$\frac{dy}{dx} = \omega(x,y)$$

if and only if $\boldsymbol{A}^{(1)}$ is tangent to the surface S_ω. Hint: review the symmetry determining equation.

Solution: Let

$$\nu_\omega(x,y) = \nu(x,y,\omega(x,y))$$

denote the above-defined function restricted to the surface $S = S_\omega$. Note, we are using (x, y) as coordinates on the surface. Thus,

$$\nu_\omega(x, y) = \eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y.$$

Also observe that the vector field $\boldsymbol{A}^{(1)}$ is tangent to the surface S if and only if the directional derivative $\boldsymbol{A}^{(1)}[y_1 - \omega(x, y)]$ vanishes on the surface S. We have

$$\boldsymbol{A}^{(1)}[y_1 - \omega(x, y)] = \nu(x, y, y_1) - (\xi\omega_x + \eta\omega_y)(x, y)$$

To restrict this function to the surface S, we make the substitution $y_1 \mapsto \omega(x, y)$ and obtain

$$\boldsymbol{A}^{(1)}[y_1 - \omega(x, y)]\Big|_{y_1=\omega(x,y)} = \eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - (\xi\omega_x + \eta\omega_y)$$

Setting the RHS to zero we obtain the symmetry determining equaton. In lecture, we showed that this equation holds if and only if $\boldsymbol{A}$ is an infinitesimal symmetry.

(3) (a) Show that ∂_y is an infinitesimal symmetry of the second-order ODE having the form

$$\frac{d^2y}{dx^2} = \omega\left(x, \frac{dy}{dx}\right). \tag{2}$$

Solution: The vector field ∂_y generates the flow

$$(\hat{x}, \hat{y}) = \Phi_t(x, y) = (x, y + t).$$

Hence, a solution $y = f(x)$ is related to a solution $\hat{y} = \hat{f}(\hat{x})$ by

$$\hat{f}(\hat{x}) = f(\hat{x}) + t.$$

Hence, the prolonged transformation is

$$\hat{y}_1 = y_1, \quad \hat{y}_2 = y_2.$$

Therefore,

$$\hat{\omega}(\hat{x}, \hat{y}, \hat{y}_1) = \omega(\hat{x}, \hat{y} - t, \hat{y}_1).$$

In order for ∂_y to be an infinitesimal symmetry we ask that

$$\hat{\omega}(\hat{x}, \hat{y}, \hat{y}_1) = \omega(\hat{x}, \hat{y}, \hat{y}_1)$$

for all t. This is true if and only if $\omega(x, y, y_1)$ is independent of y.
Here is an alternate solution. Note that the prolongation of ∂_y is just $\partial_y + 0\partial_{y_1}$. Observe that

$$[\partial_y, \partial_x + y_1\partial_y + \omega(x, y, y_1)\partial_{y_1}] = \omega_y\partial_{y_1}.$$

The RHS is proportional to $\partial_x + y_1\partial_y + \omega\partial_{y_1}$ if and only if ω does not depend on y.

(b) Show that the above ODE admits the following reduction of order:

$$\frac{dy_1}{dx} = \omega(x, y_1), \quad \frac{dy}{dx} = y_1.$$

SOLUTION: Suppose that $y_1 = g(x)$ is a solution of the first ODE, i.e., $g'(x) = \omega(x, g(x))$, and that $y = f(x)$ is a solution of the second ODE, i.e., $f'(x) = g(x)$. Then,

$$f''(x) = \omega(x, f'(x)).$$

Therefore, $y = f(x)$ is a solution of (2).

(4) Let $y = f(x)$ be a non-zero solution of the second-order homogeneous linear ODE

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0. \tag{3}$$

(a) Consult a standard ODE text and describe the usual procedure for reduction of order in such situations.

SOLUTION: Seting $v = y/f(x)$, we have

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dv}{dx}f(x) + vf'(x),\\
0 &= \frac{d^2y}{dx^2} + p(x)f(x)\frac{dv}{dx} + (p(x)f'(x) + q(x)f(x))v\\
&= \frac{d^2v}{dx^2}f(x) + (2f'(x) + p(x)f(x))\frac{dv}{dx} +\\
&\quad + (f''(x) + p(x)f'(x) + q(x)f(x))v\\
&= f(x)\left(\frac{d^2v}{dx^2} + \left(2\frac{f'(x)}{f(x)} + p(x)\right)\frac{dv}{dx}\right)
\end{aligned}$$

One then solves the equation

$$\frac{dv_1}{dx} = -\left(2\frac{f'(x)}{f(x)} + p(x)\right)v_1 \tag{4}$$

by setting

$$v_1 = g(x) = \exp\left(-\int^x p(s)ds\right)/f(x)^2. \tag{5}$$

Finally, the general solution of the 2nd order equation (3) is

$$y = f(x)\int^x g(s)ds.$$

(b) Show that $f(x)\partial_y$ is a symmetry of the above ODE.

SOLUTION: Note that $f(x)\partial_y$ is the infinitesimal generator of the flow

$$(\hat{x}, \hat{y}) = \Phi(t, x, y) = (x, y + f(x)t).$$

Calculating the prolongation, we have

$$\begin{aligned}\hat{y}_1 &= y_1 + f'(x)t,\\ \hat{y}_2 &= y_2 + f''(x)t.\end{aligned}$$

Hence, the transformed equation is given by

$$\begin{aligned}\hat{y}_2 &= y_2 - f''(x)t\\ &= p(x)y_1 + q(x)y - f''(x)t\\ &= p(x)(\hat{y}_1 - f'(x)t) + q(x)(\hat{y} - tf(x)) - f''(x)t\end{aligned}$$

Since, by assumption,

$$f''(x) = p(x)f'(x) + q(x)f(x),$$

we have

$$\hat{y}_2 = p(x)\hat{y}_1 + q(x)\hat{y}$$

Since $f(x)\partial_y$ generates a symmetry of (3) for all t, it is an infinitesimal symmetry. Here is an alternate solution. Calculating the prolongation of $\boldsymbol{A} = f(x)\partial_y$ gives

$$\begin{aligned}\boldsymbol{A}^{(1)} &= f(x)\partial_y + (\partial_x - y_1\partial_y)[f(x)]\\ &= f(x)\partial_y + f'(x)\partial_{y_1}.\end{aligned}$$

In lecture we shoed that $\boldsymbol{A}$ is an infinitesimal symmetry if and only if $\boldsymbol{A}$ is a infinitesimal conformal symmetry of $\partial_x + y_1\partial_y + (p(x)y_1 + q(x)y)\partial_{y_1}$. Calculating the bracket, we have

$$\begin{aligned}&[f(x)\partial_y + f'(x)\partial_{y_1}, \partial_x + y_1\partial_y + (p(x)y_1 + q(x)y)\partial_{y_1}] =\\ &\quad = f'(x)\partial_y + (p(x)f'(x) + q(x)f(x))\partial_{y_1} - f'(x)\partial_y - f''(x)\partial_{y_1}\\ &\quad = (p(x)f'(x) + q(x)f(x) - f''(x))\partial_{y_1} = 0.\end{aligned}$$

(c) Show that $f(x)\partial_y = \partial_v$ where $u = x,\ v = y/f(x)$, and that correspondingly the ODE in question takes the form

$$\frac{d^2v}{du^2} = \omega\left(u, \frac{dv}{du}\right).$$

Conclude by showing that the reduction of order described in question 6 is equivalent to the standard reduction of order procedure described in part (a) of the present question.

Solution: As was shown in part (a), making the above change of coordinates transforms (3) into

$$\frac{d^2v}{du^2} = -\left(2\frac{f'(x)}{f(x)} + p(x)\right)\frac{dv}{du}.$$

With $v_1 = dv/du$ the procedure we followed in part (a) is the same reduction of order procedure described in question 3.
Let us give the details. The calculations will be simpler if we write $x = u, y = vf(u)$. By the prolongation formulas,

$$\begin{aligned} y_1 &= \frac{y_u + v_1 y_v}{x_u + v_1 x_v} = vf'(u) + v_1 f(u) \\ y_2 &= \frac{(\partial_u + v_1\partial_v)[y_1]}{x_u + v_1 x_v} + \frac{x_u y_v - x_v y_u}{(x_u + v_1 x_v)^3} v_2 \\ &= (vf''(u) + v_1 f'(u) + v_1 f'(u)) + f(u)v_2 \end{aligned}$$

Substituting the above relations into $y_2 + p(x)y_1 + q(x)y = 0$ and using the assumption about $f''(x)$ gives

$$\begin{aligned} 0 = v(-p(u)f'(u) - q(u)f(u)) + 2v_1 f'(u) + f(u)v_2 + \\ +(vf'(u) + v_1 f(u))p(u) + vf(u)q(u) \\ = f(u)v_2 + v_1(2f'(u) + p(u)f(u)) \\ v_2 = (-2f'(u)/f(u) - p(u))v_1 \end{aligned}$$

(5) A hodograph transformation is a change of coordinates that reverses the roles of the dependent and independent variables. For scalar ODEs the hodograph transformation is simply

$$\hat{x} = y, \quad \hat{y} = x.$$

In particular we say that functions $y = f(x)$ and $\hat{y} = \hat{f}(\hat{x})$ are related by a hodograph transformation if

$$x = \hat{y} = \hat{f}(\hat{x}) = \hat{f}(y) = \hat{f}(f(x));$$

in other words if f and $\hat{f}$ are functional inverses.

(a) Show that second-order ODEs

$$\frac{d^2y}{dx^2} = \omega\left(x, y, \frac{dy}{dx}\right), \quad \frac{d^2\hat{y}}{d\hat{x}^2} = \hat{\omega}\left(\hat{x}, \hat{y}, \frac{d\hat{y}}{d\hat{x}}\right)$$

are related by a hodograph transformation if and only if

$$\hat{\omega}(\hat{x}, \hat{y}, \hat{y}_1) = -(\hat{y}_1)^3\,\omega(\hat{y}, \hat{x}, 1/\hat{y}_1)$$

SOLUTION: Letting $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$, and similarly $\hat{y}_1 = \frac{d\hat{y}}{d\hat{x}}$ and $\hat{y}_2 = \frac{d^2\hat{y}}{d\hat{x}^2}$, we have

$$y_1\hat{y}_1 = 1,$$
$$y_2\,\hat{y}_1 + \hat{y}_2(y_1)^2 = 0$$

Hence,

$$\hat{y}_2 = -(\hat{y}_1)^3 y_2,$$

as was to be shown.

(b) Use the hodograph transformation to solve the ODE

$$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^3.$$

SOLUTION: The transformed equation is

$$\frac{d^2\hat{y}}{d\hat{x}^2} = -1.$$

The solution is

$$\hat{y} = -\frac{1}{2}\hat{x}^2 + C_1\hat{x} + C_0.$$

In (x, y) coordinates, the above solution is described, implicitly, by the relation

$$x = -\frac{1}{2}y^2 + C_1 y + C_0.$$

Solving for y gives the explicit solution:

$$y = C_1 \pm \sqrt{C_2 - 2x},$$

where $C_2 = C_1^2 + 2C_0$.

(c) Use the above transformation law to show that the hodograph transformation relates an ODE of the form

$$\frac{d^2y}{dx^2} = \omega\left(x, \frac{dy}{dx}\right) \tag{6}$$

(see Question 6) to an autonomous ODE.

SOLUTION: Using part (a), a hodograph transformation changes (6) into

$$\frac{d^2\hat{y}}{d\hat{x}^2} = (\hat{y}_1)^3\omega\left(\hat{y}, \frac{1}{\hat{y}_1}\right). \tag{7}$$

The right hand side does not depend on $\hat{x}$; therefore, the above ODE is autonomous in $\hat{x}, \hat{y}$ coordinates.

(6) (a) Let y be a real number, and consider the function $\hat{T}_y : \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{T}_y : x \mapsto x + (y - \sin(x)), \quad x \in \mathbb{R}.$$

Fix a value of y between -1 and 1 (your choice), set $x_0 = 0$, and inductively define $x_{k+1} = \hat{T}_y(x_k)$. Write down the first few elements of the x_k sequence. Compare these numbers to $\arcsin(y)$. Report your findings. Next, fix a value of $y > 1$, and again write down the first few elements of the sequence x_k. Report your findings.

SOLUTION: We have

$$\begin{aligned}
x_1 &= x_0 + (y - \sin(x_0)) = y, \\
x_2 &= x_1 + (y - \sin(x_1)) = 2y - \sin y, \\
x_3 &= x_2 + (y - \sin(x_2)) = 3y - \sin(2y - \sin(y)), \\
x_4 &= x_3 + (y - \sin(x_3)) = 4y - \sin(3y - \sin(2y - \sin(y))).
\end{aligned}$$

By way of illustration, let's set $y = 0.5$. The sequence x_k assumes the values

$$0,\ 0.5,\ 0.520574,\ 0.523196,\ 0.523545,\ 0.523592, \ldots.$$

We see that the above sequence is drawing closer to the value $\arcsin(0.5) = 0.523599...$

Next, let's take $y = 1.5$. The sequence x_k then assumes the values

$$0,\ 1.5,\ 2.00251,\ 2.59425,\ 3.57384,\ 5.49274, \ldots$$

The latter does not appear to be converging to anything.

(b) Let $\mathcal{C}_\epsilon = \mathcal{C}([-\epsilon, \epsilon], \mathbb{R})$ and consider the operator $T : \mathcal{C}_\epsilon \to \mathcal{C}_\epsilon$ defined by

$$T[g](y) = g(y) + (y - \sin(g(y))), \quad g \in \mathcal{C}_\epsilon\,.$$

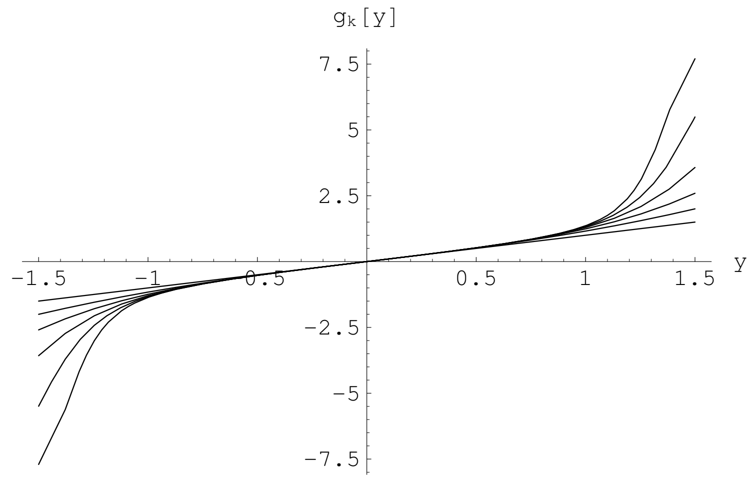
Let $g_0 = 0 \in \mathcal{C}_\epsilon$ be the zero function and inductively define $g_{k+1} = T[g_k]$. Write out the first few functions in the sequence $\{g_k\}_{k=0}^{\infty}$

SOLUTION: We have

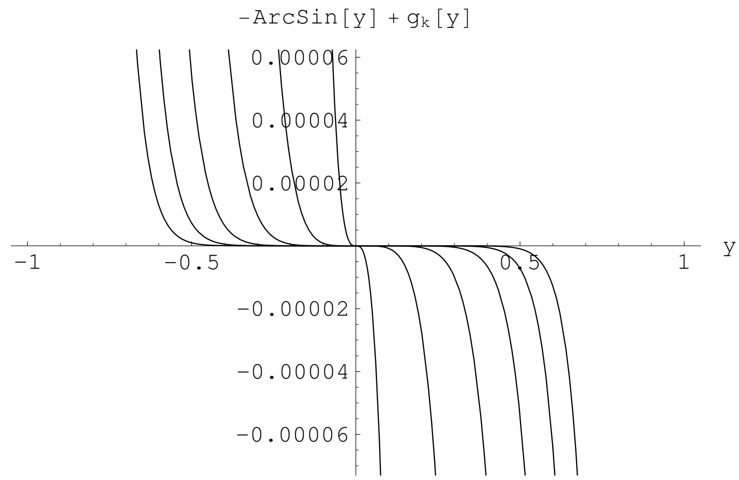
$$\begin{aligned}
g_1(y) &= y, \\
g_2(y) &= g_1(y) + (y - \sin(g_1(y))) = 2y - \sin y, \\
g_3(y) &= g_2(y) + (y - \sin(g_2(y))) = 3y - \sin(2y - \sin(y)), \\
g_4(y) &= g_3(y) + (y - \sin(g_3(y))) = 4y - \sin(3y - \sin(2y - \sin(y))).
\end{aligned}$$

Note the great similarity to the formulas in part (a).

(c) Use Maple to plot the first few elements of the sequence $g_k(y)$ over the range $-1.5 \leq y \leq 1.5$. Describe the apparent convergence properties. Use Maple to plot the difference $g_k(y) - \arcsin(y)$ over the range $-1 \leq y \leq 1$ and, again, describe the apparent convergence properties.



SOLUTION:
We see that the functions $g_k(y)$ appear to exhibit convergent behaviour over the range $-1 < y < 1$, and divergent behaviour outside that range.



We see that the functions $g_k(y)$ appear to furnish an approximation of $\arcsin(y)$. The approximation improves as k gets larger, and gets worse for y further away from $y_0 = 0$.

(d) Let K be a real number strictly between 0 and 1. Set $\delta = \sqrt{2K}$. Show that for $x_1, x_2 \in B_\delta(0)$ and for every y we have

$$|\hat{T}_y(x_1) - \hat{T}_y(x_2)| \leq K|x_1 - x_2|.$$

Hint: use the mean value theorem and the fact (explain) that

$$\frac{x^2}{2} > 1 - \cos(x).$$

Solution: Let x_1, x_2, both within a distance of δ of 0, be given. Hence, by the mean value theorem, there exists a ξ between x_1, x_2 such that

$$\hat{T}_y(x_1) - \hat{T}_y(x_2) = x_1 - x_2 + (\sin(x_2) - \sin(x_1)) = (1 - \cos(\xi))(x_1 - x_2).$$

Note that, since

$$1 - \cos(x) = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \dots$$

is, for all x, an alternating series, we have that

$$\frac{x^2}{2!} > 1 - \cos(x)$$

for all x. Also note that ξ is within a distance δ of 0. Hence,

$$|\hat{T}_y(x_1) - \hat{T}_y(x_2)| \leq \frac{\xi^2}{2}|x_1 - x_2| \leq \frac{\delta^2}{2}|x_1 - x_2| \leq K|x_1 - x_2|,$$

as was to be shown.

(e) Set $\epsilon = \delta(1 - K)$. Let $x \in B_\delta(0)$ and $y \in B_\epsilon(0)$, and set $\hat{x} = \hat{T}_y(x)$. Prove that $\hat{x} \in B_\delta(0)$ also. Hint, you will need to show that

$$|x - \sin(x)| \leq K|x|.$$

Solution: Let $x, y, \hat{x}$ be as above. Applying the result in part (d) with $x_1 = x, x_2 = 0$, we obtain that

$$|x - \sin(x)| \leq K|x|.$$

Hence,

$$|\hat{x} - 0| = |x - \sin(x) + y| \leq |x - \sin x| + |y| \leq K\delta + \epsilon = \delta,$$

as was to be shown.

(f) Fix $\delta > 0$ and let $\mathcal{C}_{\epsilon,\delta} \subset \mathcal{C}_\epsilon$ be the closed subset consisting of functions whose absolute value is bounded by δ, i.e., the closed ball of radius δ around the zero function in $\mathcal{C}_\epsilon$. With T as above, show that $T : \mathcal{C}_{\epsilon,\delta} \to \mathcal{C}_{\epsilon,\delta}$ is a well defined operator Hint: use part (e).

Solution: Let $g(y) \in \mathcal{F}$ be a continuous function as per the above definition. Since $T[g](y)$ is obtained from $g(y)$ by addition and by composition with continuous functions, $T[g]$ is continuous also. Thus, $T[g] \in \mathcal{C}(B_\epsilon(0), \mathbb{R})$. It remains to show that $T[g] \in \mathcal{F}$. Let $y \in B_\epsilon(0)$ be given, set $x = g(y)$, $\hat{x} = T[g](y)$. Note that, by definition,

$$\hat{x} = \hat{T}_y(x).$$

By assumption, $|x| \leq \delta$. Hence, by part (e), $|\hat{x}| \leq \delta$ also. It follows that $|T[g](y) - 0|$ is bounded above by δ, as was to be shown.

(g) Show that $T : \mathcal{C}_{\epsilon,\delta} \to \mathcal{C}_{\epsilon,\delta}$ is a contraction operator with Lipschitz constant K. Hint: use part (d).
SOLUTION: Let $g_1, g_2 \in \mathcal{F}$ be given. Let $y \in B_\epsilon(0)$ be given, and set

$$\begin{aligned} x_1 &= g_1(y), \quad x_2 = g_2(y) \\ \hat{g}_1 &= T[g_1], \quad \hat{g}_2 = T[g_2] \\ \hat{x}_1 &= \hat{g}_1(y) = \hat{T}_y(x_1), \\ \hat{x}_2 &= \hat{g}_2(y) = \hat{T}_y(x_2). \end{aligned}$$

We showed in part (d) that

$$|\hat{x}_1 - \hat{x}_2| \le K|x_1 - x_2|.$$

Hence, for all $y \in B_\epsilon(0)$ we have

$$|\hat{g}_1(y) - \hat{g}_2(y)| \le K|g_1(y) - g_2(y)| \le Kd(g_1, g_2).$$

Since the above holds for all y, we have that

$$d(\hat{g}_1, \hat{g}_2) = \sup_y |\hat{g}_1(y) - \hat{g}_2(y)| \le Kd(g_1, g_2),$$

as was to be shown.

(h) Use the fixed point theorem to conclude that the function $g(y) = \arcsin(y),\ |y| \le \epsilon$ is the unique fixed point of the operator $T : \mathcal{C}_{\epsilon,\delta} \to \mathcal{C}_{\epsilon,\delta}$
SOLUTION: Note that $T[g] = g$; this is essentially the same statement as $\sin(g(y)) = y$. Note that $\mathcal{F}$ is the set of all functions in $\mathcal{C}(B_\epsilon(0), \mathbb{R})$ within a distance δ of the zero function. Thus, by 2(a), $\mathcal{F}$ is a complete metric space. We've already shown that $T : \mathcal{F} \to \mathcal{F}$ is a contracting operator. Therefore, by the fixed point theorem, T possesses a unique fixed point; that fixed point must be the $g(y) = \arcsin(y)$ function.