Assignment 5, Solutions

(1) Let A be an $n \times n$ real matrix and

$$\|A\| = \sup\left\{\frac{\|Au\|}{\|u\|} : u \in \mathbb{R}^n, u \neq 0\right\}$$

the operator norm, relative to the Euclidean norm

$$\|u\| = \sqrt{u \cdot u}, \quad u \in \mathbb{R}^n.$$

(a) Prove that

$$\begin{aligned} &\|A + B\| \leq \|A\| + \|B\|, \quad A, B \in \operatorname{Mat}_{n\times n} \mathbb{R} \\ &\|AB\| \leq \|A\| \, \|B\|. \end{aligned}$$

Solution: The solution is available in most linear algebra text books. See, for example, Section 7.2 of D. Poole's Linear Algebra.

(b) Let $S = S^t$ be a symmetric matrix. Prove that

$$\|S\| = \max\{|\lambda| : \det(S - \lambda I) = 0\}$$

Solution: Since S is symmetric, there exists an orthogonal basis of eigenvectors. Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ be such a basis. Thus,

$$S\boldsymbol{u}_i = \lambda_i \boldsymbol{u}_i,$$

where $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues, and where

$$\begin{aligned} \boldsymbol{u}_i \cdot \boldsymbol{u}_i &= 1, \quad i = 1, \ldots, n; \\ \boldsymbol{u}_i \cdot \boldsymbol{u}_j &= 0, \quad i \neq j. \end{aligned}$$

Let a non-zero $\boldsymbol{u} \in \mathbb{R}^n$ be given. Write

$$\boldsymbol{u} = c_1 \boldsymbol{u}_1 + \cdots c_n \boldsymbol{u}_n, \quad c_i = \boldsymbol{u} \cdot \boldsymbol{u}_i,$$

and note that

$$\begin{aligned} S\boldsymbol{u} &= S(c_1 \boldsymbol{u}_1 + \cdots c_n \boldsymbol{u}_n) \\ &= c_1 \lambda_1 \boldsymbol{u}_1 + \cdots + c_n \lambda_n \boldsymbol{u}_n. \end{aligned}$$

Without loss of generality (rearrange the order of the eigenvalues, as appropriate) assume that $|\lambda_1| = \max_i |\lambda_i|$. Since $\boldsymbol{u} \cdot \boldsymbol{u} = c_1^2 + \cdots + c_n^2$, we have

$$\begin{aligned} \left(\frac{\|Su\|}{\|u\|}\right)^2 &= |\lambda_1|^2 \left(\frac{c_1^2 + (\lambda_2/\lambda_1)^2 c_2^2 + \cdots + (\lambda_n/\lambda_1)^2 c_n^2}{c_1^2 + \cdots + c_n^2}\right) \\ &\leq |\lambda_1|^2. \end{aligned}$$

Hence, $\|S\| \leq \max_\lambda |\lambda|$ where λ indexes the set of eigenvalues of S. On the other hand, we have

$$\begin{aligned} \frac{\|Su_1\|}{\|u_1\|} &= \frac{\|\lambda_1 u_1\|}{\|u_1\|} \\ &= |\lambda_1|. \end{aligned}$$

The above calculation shows that, indeed $\|S\| = |\lambda_1|$, where $|\lambda_1| = \max_\lambda |\lambda|$ by assumption.

(2) (a) Find the Jordan canonical form J of the matrix

$$A = \begin{pmatrix} -1 & 3 & 0 \\ -3 & -1 & 3 \\ 0 & 3 & -1 \end{pmatrix}.$$

SOLUTION: The characteristic polynomial is

$$p_A(\lambda) = -(\lambda+1)^3$$

Therefore there is just one, triple eigenvalue, namely $\lambda = -1$. Consideration of the nullspace of $A + I$ gives an eigenvector, namely

$$\boldsymbol{u}_1 = \mathbf{e}_1 + \mathbf{e}_3, \quad A\boldsymbol{u}_1 = -\boldsymbol{u}_1.$$

The nullspace of $(A+I)^2$ is 2 dimensional; a basis is given by $\boldsymbol{u}_1$ and by $\mathbf{e}_1$. Observe that

$$A\mathbf{e}_1 = -\mathbf{e}_1 + 3\boldsymbol{u}_1.$$

Hence, by setting $\boldsymbol{u}_2 = (1/3)\mathbf{e}_1$ we have

$$A\boldsymbol{u}_2 = -\boldsymbol{u}_2 + \boldsymbol{u}_1.$$

Finally, $(A+I)^3 = 0$, hence to complete the basis we can take an arbitrary $\boldsymbol{u}_3$ such that $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ is linearly independent and such that

$$A\boldsymbol{u}_3 = -\boldsymbol{u}_3 + \boldsymbol{u}_2.$$

One way this can be accomplished is to take $\boldsymbol{u}_3 = (1/9)\mathbf{e}_3$. Letting $P = [\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3]$ be the corresponding change of basis matrix, we obtain

$$N = P^{-1}AP = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

as the Jordan form for A.

(b) Calculate $\exp(tJ)$.
SOLUTION: Using the identities derived in lecture, we have

$$\begin{aligned}\exp(tJ) &= \exp(-tI + tN) = e^{-t}\exp tN \\ &= e^{-t}\begin{bmatrix}1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1\end{bmatrix}\end{aligned}$$

(c) What is the general solution of the linear ODE

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}.$$

SOLUTION: The flow $\Phi(t, \boldsymbol{x}) = \exp(tA)\boldsymbol{x}$ generated by A is given by

$$\begin{aligned}\exp(tA) &= P\exp(tJ)P^{-1} \\ &= e^{-t}\begin{bmatrix}1-(9/2)t^2 & 3t & (9/2)t^2 \\ -3t & 1 & 3t \\ -(9/2)t^2 & 3t & 1+(9/2)t^2\end{bmatrix}\end{aligned}$$

(3) For a complex number $z = a + \mathrm{i}\,b$, let

$$z^{\mathrm{R}} = \begin{pmatrix}a & -b \\ b & a\end{pmatrix}.$$

If A is an $n \times n$ complex matrix, let A^{R} denote the $(2n) \times (2n)$ matrix obtained by replacing each entry $A_{ij} \in \mathbb{C}$ with A_{ij}^{R}.

(a) Let A, B be $n \times n$ complex matrices. Prove that $(AB)^{\mathrm{R}} = A^{\mathrm{R}}B^{\mathrm{R}}$
SOLUTION: This follows from the validity of block multiplication of matrices.

(b) Let $a, b \in \mathbb{R}$. Calculate $\exp(tB)$, where

$$B = \begin{pmatrix}a & -b & 1 & 0 \\ b & a & 0 & 1 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a\end{pmatrix}.$$

Hint: find a 2×2 complex matrix A such that $B = A^{\mathrm{R}}$.

SOLUTION: Set $\lambda = a + ib$ and note that $B = A^{\mathrm{R}}$, where

$$A = \begin{bmatrix}\lambda & 1 \\ 0 & \lambda\end{bmatrix}.$$

Observe that

$$\exp(tA) = e^{t\lambda}\begin{bmatrix}1 & t \\ 0 & 1\end{bmatrix}$$

Hence,

$$\exp(tB) = \exp(tA)^{\mathrm{R}} = e^{ta}\begin{bmatrix} \cos(bt) & -\sin(bt) & t\cos(bt) & -t\sin(bt) \\ \sin(bt) & \cos(bt) & t\sin(bt) & t\cos(bt) \\ 0 & 0 & \cos(bt) & -\sin(bt) \\ 0 & 0 & \sin(bt) & \cos(bt) \end{bmatrix}.$$

(4) In this exercise we will use Picard iteration to obtain the solution to the linear ODE:

$$\dot{x} = -y, \quad \dot{y} = x.$$

(a) Let $\mathbf{V} : \mathbb{R}^2 \to \mathbb{R}^2$ denote the above vector field function, i.e.

$$\mathbf{V}(x, y) = (-y, x).$$

What is the flow generated by $\mathbf{V}$?
Solution:

$$\Phi(t, x, y) = (x\cos(t) - y\sin(t), x\sin(t) + y\cos(t)).$$

(b) Let $D = B_1(0) \subset \mathbb{R}^2$ denote the closed unit disk in $\mathbb{R}^2$, and let $I = [-\frac{1}{2}, \frac{1}{2}] \subset \mathbb{R}$. Let

$$\mathcal{F} = \{\phi \in \mathcal{C}^0(I \times D, \mathbb{R}^2) : \phi(0, x, y) = (x, y)\}.$$

Define the operator $\mathcal{P} : \mathcal{F} \to \mathcal{F}$ by

$$\mathcal{P}[\phi](t, x, y) = (x, y) + \int_0^t \mathbf{V}(\phi(s, x, y))ds, \quad \phi \in \mathcal{F}$$

Define $\phi_0(t, x, y) = (x, y)$, and then $\phi_{k+1} = \mathcal{P}[\phi_k]$. Explicitly determine $\phi_1, \phi_2, \phi_3, \phi_4$.
Solution:

$$\phi_1(t, x, y) = \begin{bmatrix} -ty \\ tx \end{bmatrix},$$

$$\phi_2(t, x, y) = \begin{bmatrix} -ty - x\frac{t^2}{2} \\ tx - y\frac{t^2}{2} \end{bmatrix} = \begin{bmatrix} -\frac{t^2}{2} & -t \\ t & -\frac{t^2}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\phi_3(t, x, y) = \begin{bmatrix} -ty - x\frac{t^2}{2} + y\frac{t^3}{3!} \\ tx - y\frac{t^2}{2} - x\frac{t^3}{3!} \end{bmatrix} = \begin{bmatrix} -\frac{t^2}{2} & -t + \frac{t^3}{3!} \\ t - \frac{t^3}{3!} & -\frac{t^2}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

More generally,

$$\phi_k(t, x, y) = \left(At + A^2\frac{t^2}{2!} + \ldots + A^k\frac{t^k}{k!}\right)\begin{bmatrix} x \\ y \end{bmatrix}.$$

(c) Prove that the sequence of functions $\{\phi_k\}_{k=0}^{\infty}$ converges uniformly to the flow determined in part (a). Hint: it's possible to write $\phi_k(t,x,y)$ using the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and its various powers. As well, in proving convergence, it may be useful to identify $\mathbb{R}^2$ with $\mathbb{C}$ and to re-express the matrix A as the imaginary number i.

Solution: Note that

$$\phi_k(t,\boldsymbol{u}) \begin{bmatrix} T_k[\cos(t)] & T_k[-\sin(t)] \\ T_k[\sin(t)] & T_k[\cos(t)] \end{bmatrix} \boldsymbol{u}, \quad \boldsymbol{u} = \begin{bmatrix} x \\ y \end{bmatrix} \in D, \quad t \in I,$$

where T_k denotes the kth Taylor polynomial in t. Also note that the Taylor polynomials for $\sin(t)$ and $\cos(t)$ converge uniformly to $\sin(t)$ and $\cos(t)$, respectively, for $|t| \leq \frac{1}{2}$ (One can use the Weierstrass M-test to show this).

Let us set

$$a_k(t) = \cos(t) - T_k[\cos(t)], \quad b_k(t) = \sin(t) - T_k[\sin(t)].$$

We are trying to prove that

$$\begin{bmatrix} a_k(t) & -b_k(t) \\ b_k(t) & a_k(t) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad t \in I, \quad \begin{bmatrix} x \\ y \end{bmatrix} \in D$$

converges uniformly to $(0,0)$ as $k \to \infty$. In doing this, it will be convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$ and to write

$$w_k(t) = a_k(t) + ib_k(t), \quad z = x + iy.$$

In this reformulation, we are trying to prove that the complex sequence

$$w_k(t)z, \quad |t| \leq \frac{1}{2}, \quad |z| \leq 1,$$

converges uniformly to $0 \in \mathbb{C}$. However,

$$|w_k(t)z| = |w_k(t)||z| \leq |w_k(t)| = \sqrt{a_k(t)^2 + b_k(t)^2}.$$

Since $a_k(t), b_k(t) \to 0$ uniformly, it follows that

$$|w_k(t)| \to 0$$

uniformly as well.

(d) Prove that $\mathbf{V} : \mathbb{R}^2 \to \mathbb{R}^2$ is a Lipschitz function with Lipschitz constant 1.

Solution: Let $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$ be given. As above, we identify $\mathbf{V}(x_1, y_1)$ with the complex number iz_1 and $\mathbf{V}(x_2, y_2)$ with the complex number iz_2. We are asked to shown that

$$|iz_1 - iz_2| \leq |z_1 - z_2|.$$

However, using the complex number norm behaves very nicely under multiplication. Indeed

$$|iz_1 - iz_2| = |i||z_1 - z_2| = |z_1 - z_2|;$$

and we are done.

(e) Prove explicitly that $\mathcal{P}$ is a contraction operator and that ϕ, the flow generated by $\mathbf{V}$ is its unique fixed point.

SOLUTION: Note that since the domain of $\mathbf{V}$ is all of $\mathbb{R}^2$, there is no difficulty in defining $\mathcal{P}$ as an operator on the function space $\mathcal{C}(I \times D, \mathbb{R}^2)$. Evidently, for a continuous function $\phi(t, x, y)$ we have

$$\mathcal{P}[\phi](0, x, y) = (x, y).$$

Hence $\mathcal{P} : \mathcal{F} \to \mathcal{F}$ is a well-defined operator.

Next, let $\phi_1, \phi_2 \in \mathcal{F}$ be given, and set

$$\hat{\phi}_1 = \mathcal{P}[\phi_1], \quad \hat{\phi}_2 = \mathcal{P}[\phi_2].$$

Let $t \in I$ and $\boldsymbol{u} \in D$ be given. We have

$$\begin{aligned} \|\hat{\phi}_1(t, \boldsymbol{u}) - \hat{\phi}_2(t, \boldsymbol{u})\| &= \| \int_0^t A\phi_1(s, \boldsymbol{u})\, ds - \int_0^t A\phi_2(s, \boldsymbol{u})\, ds\| \\ &= \| \int_0^t A(\phi_1(s, \boldsymbol{u}) - \phi_2(s, \boldsymbol{u}))\, ds\| \\ &\leq \int_0^t \|A(\phi_1(s, \boldsymbol{u}) - \phi_2(s, \boldsymbol{u}))\| ds \\ &= \int_0^t \|\phi_1(s, \boldsymbol{u}) - \phi_2(s, \boldsymbol{u})\| ds; \end{aligned}$$

the last equality follows by the same argument used in part (d). Next, let

$$M = d(\phi_1, \phi_2) = \sup_{t, \boldsymbol{u}} \|\phi_1(t, \boldsymbol{u}) - \phi_2(t, \boldsymbol{u})\|.$$

Hence, for all $t \in I$ and all $u \in D$, we have

$$\|\hat{\phi}_1(t, \boldsymbol{u}) - \hat{\phi}_2(t, \boldsymbol{u})\| \leq \int_0^t M\, ds = Mt \leq \frac{1}{2} M.$$

Hence,

$$d(\hat{\phi}_1, \hat{\phi}_2) = \sup_{t, \boldsymbol{u}} \|\hat{\phi}_1(t, \boldsymbol{u}) - \hat{\phi}_2(t, \boldsymbol{u})\| \leq \frac{1}{2}\, d(\phi_1, \phi_2),$$

as was to be shown.

In lecture we showed that $\mathcal{C}(I \times D, \mathbb{R}^2)$ is a complete metric space. The same argument shows that $\mathcal{F}$ is a complete metric space. To wit, a sequence of functions $\phi_k(t, x, y) \in \mathcal{F}$ (i.e., each ϕ_k is continuous, and furthermore $\phi_k(0, x, y) = (x, y)$) that is a Cauchy sequence relative to the supremum norm, converges uniformly to some continuous limit $\phi(t,x,y)$. However, since uniform convergence implies point convergence, we must have $\phi(0,x,y) = (x,y)$, as well.

Since T is a contraction operator (with contraction constant 1/2) on the complete metric space $\mathcal{F}$, it has a unique fixed point, by the Fixed Point Theorem. However, the conditions

$$\frac{\partial \Phi}{\partial t}(t,x,y) = \mathbf{V}(\Phi(t,x,y)), \quad \Phi(0,x,y) = (x,y), \quad t \in I, \quad (x,y) \in D,$$

are equivalent to the conditions

$$\phi(t,\boldsymbol{u}) = \mathcal{P}[\phi](t,\boldsymbol{u}) = \boldsymbol{u} + \int_0^t \mathbf{V}(\phi(s,\boldsymbol{u}))\, ds, \quad \phi(0,\boldsymbol{u}) = \boldsymbol{u},$$

where

$$\boldsymbol{u} = (x,y) \in D,\ t \in I.$$

Therefore, $\phi(t,x,y)$ is the unique fixed point of the Picard operator $\mathcal{P}$.

Note: in particular, the Fixed Point Theorem gives us another proof that the above approximations $\phi_k(t,x,y)$ converge uniformly to $\phi(t,x,y)$.

(5) (a) Find the general solution of the time-dependent linear ODE

$$\dot{x} = -\cos(t)y, \quad \dot{y} = \cos(t)x.$$

Hint: decouple the equations.

SOLUTION: Let us rewrite the above ODE in matrix form as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \cos(t) A \begin{bmatrix} x \\ y \end{bmatrix},$$

where A is the rotation generator matrix used in question 4. The eigenvalues of A are $\pm i$ with eigenvectors $\mathbf{e}_1 \mp i\mathbf{e}_2$, respectively. Let us therefore introduce complex variables

$$u = x + iy, \quad v = x - iy$$

and rewrite the above ODE (formally) in terms of these variables. We have

$$\begin{aligned} \dot{u} &= \dot{x} + i\dot{y} = i\cos(t)(x+iy) = i\cos(t)u \\ \dot{v} &= \dot{x} - i\dot{y} = -i\cos(t)(x-iy) = -i\cos(t)v \end{aligned}$$

Formally solving the above complex ODEs using separation of variables gives

$$\begin{aligned} u &= \exp(i\sin(t))u_0 \\ v &= \exp(-i\sin(t))v_0 \end{aligned}$$

Switching back to x, y coordinates we obtain the following real solution

$$x = \Re(u) = \cos(\sin(t))x_0 - \sin(\sin(t))y_0,$$
$$y = \Im(u) = \sin(\sin(t))x_0 + \cos(\sin(t))y_0.$$

This can easily be verified to be the desired solution, in flow, form of the given ODE.

(b) Autonomize the above ODE and give the flow generated by the corresponding 3-dimensional vector field.

SOLUTION: We introduce an auxilliary variable τ and rewrite our ODE as

$$\dot{x} = -\cos(\tau)y,$$
$$\dot{y} = \cos(\tau)x,$$
$$\dot{\tau} = 1$$

The flow is given by the above general solution, appropriately written. Indeed, the desired 3D flow is just

$$\Phi_t(x, y, \tau) =$$
$$(\cos(\sin(t+\tau))x - \sin(\sin(t+\tau))y, \sin(\sin(t+\tau))x + \cos(\sin(t+\tau))y, \tau + t).$$