Paul Baird-Smith, Alyssa Epstein, Kristen Flint, and Steven J. Miller
The Generalized Zeckendorf Game,
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Abstract

Zeckendorf proved that every positive integer n can be written uniquely as the sum of non-adjacent Fibonacci numbers; a similar result, though with a different notion of a legal decomposition, holds for many other sequences. We use these decompositions to construct a two-player game, which can be completely analyzed for linear recurrence relations of the form $G_n = \sum_{i=1}^{k} cG_{n-i}$ for a fixed positive integer c ($c = k-1 = 1$ gives the Fibonaccis). Given a fixed integer n and an initial decomposition of $n = nG_1$, the two players alternate by using moves related to the recurrence relation, and whoever moves last wins. The game always terminates in the Zeckendorf decomposition, though depending on the choice of moves the length of the game and the winner can vary. We find upper and lower bounds on the number of moves possible; for the Fibonacci game the upper bound is on the order of $n \log n$, and for other games we obtain a bound growing linearly with n. For the Fibonacci game, Player 2 has the winning strategy for all $n > 2$. If Player 2 makes a mistake on his first move, however, Player 1 has the winning strategy instead. Interestingly, the proof of both of these claims is non-constructive.