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Linear Independence of Infinite Products Generated by the Lucas Numbers,
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Abstract

The purpose of this paper is to give linear independence results for the infinite products

$$\prod_{n=1}^{\infty}\left(1+\frac{q^n z}{q^{2n}+1}\right),$$

where q ($|q|>1$) and z are algebraic integers with suitable conditions. As an application, we derive that the ten numbers

$$1, \quad \sum_{n=1}^{\infty}\frac{1}{L_{2n}}, \quad \prod_{n=1}^{\infty}\left(1\pm\frac{1}{L_{2n}}\right), \quad \prod_{n=1}^{\infty}\left(1\pm\frac{2}{L_{2n}}\right),$$

$$\prod_{n=1}^{\infty}\left(1\pm\frac{\Phi}{L_{2n}}\right), \quad \prod_{n=1}^{\infty}\left(1\pm\frac{\Phi^{-1}}{L_{2n}}\right)$$

are linearly independent over $\mathbb{Q}(\sqrt{5})$, where L_{2n} is the $2n$-th Lucas number and Φ is the golden ratio, and that

$$\sum_{n=1}^{\infty}\frac{1}{L_{2n}+a}\notin\mathbb{Q}(\sqrt{5})$$

for any $a=\pm1,\pm2,\pm\Phi,\pm\Phi^{-1}$.