Ruoci Li, Xiaonan Li, Steven J. Miller, Clayton Mizgerd, Chenyang Sun, Dong Xia, and Zhyi Zhou
Deterministic Zeckendorf Games,
Fibonacci Quart. **58** (2020), no. 5, 152–160.

Abstract

Zeckendorf [Ze] proved that every positive integer can be written uniquely as the sum of non-adjacent Fibonacci numbers. We further explore a two-player Zeckendorf game introduced in [BEFM1, BEFM2]: Given a fixed integer n and an initial decomposition of $n = nF_1$, players alternate using moves related to the recurrence relation $F_{n+1} = F_n + F_{n_1}$, and the last player to move wins. We improve the upper bound on the number of moves possible and show that it is of the same order in n as the lower bound; this is an improvement by a logarithm over previous work. The new upper bound is $3n - 3Z(n) - IZ(n) + 1$, and the existing lower bound is sharp at $n - Z(n)$ moves, where $Z(n)$ is the number of terms in the Zeckendorf decomposition of n and $IZ(n)$ is the sum of indices in the same Zeckendorf decomposition of n. We also studied four deterministic variants of the game, where there was a fixed order on which available move one takes: Combine Largest, Split Largest, Combine Smallest and Split Smallest. We prove that Combine Largest and Split Largest realize the lower bound. Split Smallest has the largest number of moves over all possible games, and is close to the new upper bound. For Combine Split games, the number of moves grows linearly with n.