Anna Cusenza, Aidan Dunkelberg, Kate Huffman, Dianhui Ke, Micah McClatchey, Steven J. Miller, Clayton Mizgerd, Vashisth Tiwari, Jingkai Ye, and Xiaoyan Zheng
Bounds on Zeckendorf Games,
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Abstract

Zeckendorf proved that every positive integer n can be written uniquely as the sum of nonadjacent Fibonacci numbers. We use this decomposition to construct a two-player game. Given a fixed integer n and an initial decomposition of $n = nF_1$, the two players alternate by using moves related to the recurrence relation $F_{n+1} = F_n + F_{n-1}$, and whoever moves last wins. The game always terminates in the Zeckendorf decomposition; depending on the choice of moves, the length of the game and the winner can vary, although for $n \geq 2$ there is a nonconstructive proof that Player 2 has a winning strategy.

Initially, the lower bound of the length of a game was order n (and known to be sharp), whereas the upper bound was of size $n \log n$. Recent work decreased the upper bound to size n, but with a larger constant than was conjectured. We improve the upper bound and obtain the sharp bound of $\frac{\sqrt{5}+3}{2}n - IZ(n) - \frac{1+\sqrt{5}}{2}Z(n)$, which is of order n as $Z(n)$ is the number of terms in the Zeckendorf decomposition of n and $IZ(n)$ is the sum of indices in the Zeckendorf decomposition of n (which are at most of sizes $\log n$ and $\log^2 n$, respectively). We also introduce a greedy algorithm that realizes the upper bound, and show that the longest game on any n is achieved by applying splitting moves whenever possible.