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Sums Related to the Fibonacci Sequence,
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Abstract

We investigate sums associated with the Fibonacci sequence F_n and the golden ratio ϕ. In particular, we study the sums $G(k) = \sum_{n=1}^{\infty} n^k/F_n$ and $H(k) = \sqrt{5} \cdot \mathrm{Li}_{-k}(1/\phi) = \sum_{n=1}^{\infty} n^k \sqrt{5}/\phi^n$. These sums generalize the reciprocal Fibonacci constant $\psi = G(0)$. We prove the asymptotic equivalence $G(k) \sim H(k)$, and moreover, $G(k)/H(k) = 1 + 1/5^{k+1} + O((\log \phi/\pi)^{k+1})$ as $k \to \infty$. We express $G(k) - H(k)$ as an alternating series, allowing us to compute values of these sums to high precision, and to prove that $G(k) > H(k)$ if and only if $k \geq 2$. We also generalize the results to their Lucas sequence analogues. As a tool, we establish a widely applicable explicit bound for polylogarithms of negative integer order.

We find explicit bounds for the integer sequences $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ defined by
$H(k)/\sqrt{5} = \mathrm{Li}_{-k}(1/\phi) = A_k + B_k\phi$. We also prove several results concerning the multiplicative structure of A_k and B_k. We show that $\{A_k \pmod{m}\}$ and $\{B_k \pmod{m}\}$ are periodic for every natural number m, and that the period is a divisor of $\lambda(m)$, where λ denotes the Carmichael function.