Steven J. Miller, Eliel Sosis, and Jingkai Ye
Winning Strategies for Generalized Zeckendorf Games,
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Abstract

Zeckendorf proved that every positive integer n can be written uniquely as the sum of non-adjacent Fibonacci numbers; a similar result holds for other positive linear recurrence sequences. These legal decompositions can be used to construct a game that starts with a fixed integer n, and players take turns using moves relating to a given recurrence relation. The game eventually terminates in a unique legal decomposition, and the player who makes the final move wins.

For the Fibonacci game, Player 2 has the winning strategy for all $n > 2$. We give a non-constructive proof that for the two-player (c, k)-nacci game, for all k and sufficiently large n, Player 1 has a winning strategy when c is even and Player 2 has a winning strategy when c is odd. Interestingly, the player with the winning strategy can make a mistake as early as the $c+1$ turn, in which case the other player gains the winning strategy. Furthermore, we proved that for the (c, k)-nacci game with players $p \geq c+2$, no player has a winning strategy for any $n \geq 3c^2 + 6c + 3$. We find a stricter lower boundary, $n \geq 7$, in the case of the three-player $(1, 2)$-nacci game. Then we extend the result from the multiplayer game to multialliance games, showing which alliance has a winning strategy or when no winning strategy exists for some special cases of multialliance games.