# LINEAR RECURSION AND FIBONACCI SEQUENCES

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Fibonacci Association

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# INTRODUCTION

These eight lessons on Linear Recursion as related to Fibonacci sequences appeared originally in successive issues of the Fibonacci Quarterly beginning with the issue of October 1968 and ending with the issue of April 1970. Apart from pagination, the correction of a few errors, and the addition of problems to some of the lessons, the material has not been changed.

The purpose of producing this booklet is to provide individuals and groups who may not have access to all these issues of the Fibonacci Quarterly the opportunity to pursue in some depth a fascinating bit of mathematics which places the Fibonacci sequences in a broader and more intelligible context.

Brother Alfred Brousseau

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# RECURRING SEQUENCES—LESSON 1

The Fibonacci Quarterly has been publishing an abundance of material over the past five years dealing in the main with the Fibonacci sequence and its relatives. Basic to the entire undertaking is the concept of RECURRING SEQUENCE. In view of this fact, a series of some eight lessons has been prepared covering this topic. In line with the word "lesson," examples of principles will be worked out in the articles and a number of "problems" will be included for the purpose of providing "exercise" in the material presented. Answers to these problems will be included on another page so that people may be able to check their work against them.

In this first lesson, the idea of sequence and recursion relation will be considered in a general way. A sequence is an ordered set of quantities. The sequence is finite if the set of quantities terminates; it is infinite if it does not. The prototype of all sequences is the sequence of positive integers: 1, 2, 3, 4, 5, ⋯. Other sequences, some quite familiar, are the following:

1, 3, 5, 7, 9, 11, 13, ⋯
2, 4, 6, 8, 10, 12, 14, 16, ⋯
1, 2, 4, 8, 16, 32, 64, ⋯
2, 6, 18, 54, 162, 486, ⋯
1, 2, 6, 24, 120, 720, 5040, 40320, ⋯
1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ⋯
1, 4, 9, 16, 25, 36, 49, 64, ⋯
1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, ⋯

For convenience of reference, the terms of sequences can be identified by the following notation: $a_1, a_2, a_3, a_4, a_5, \cdots, a_n, \cdots$. One of the common ways of providing a compact representation of a sequence is to specify a formula for the $n^{th}$ term. For the positive integers, $a_n = n$; for the odd integers 1, 3, 5, 7, ⋯, $a_n = 2n - 1$; for the even integers 2, 4, 6, 8, ⋯

$a_n = 2n$. The $n^{th}$ terms of the remaining sequences given above are listed herewith.

$1, 2, 4, 8, 16, 32, \cdots, a_n = 2^{n-1}$
$2, 6, 18, 54, 162, 486, \cdots, a_n = 2 \cdot 3^{n-1}$
$1, 2, 6, 24, 120, \cdots, a_n = n!$
$1, 3, 6, 10, 15, 21, 28, \cdots, a_n = n(n+1)/2$
$1, 4, 9, 16, 25, 36, \cdots, a_n = n^2$
$1, 1/2, 1/3, 1/4, \cdots, a_n = 1/n$ .

There is, however, a second way of specifying sequences and that is the recursion approach. The word recursion derives from recur and indicates that something is happening over and over. When in a sequence, there is an operation which enables us to find a subsequent term by using previous terms according to some well-defined method, we have what can be termed a recursion sequence. Again, the prototype is the sequence of positive integers which is completely specified by giving the first term $a_1 = 1$ and stating the recursion relation

$$a_{n+1} = a_n + 1 \quad .$$

This is the general pattern for a recursion sequence; one or more initial terms must be specified; then an operation (or operations) is set down which enables one to generate any other term of the sequence.

Going once more to some of our previous sequences, the recursion representations are as follows:

$1, 3, 5, 7, \cdots, \ a_1 = 1; \quad a_{n+1} = a_n + 2$ .
$2, 4, 6, 8, \cdots, \ a_1 = 2; \quad a_{n+1} = a_n + 2$.
$1, 2, 4, 8, 16, \cdots, \ a_1 = 1; \quad a_{n+1} = 2a_n$ .
$2, 6, 18, 54, 162, \cdots, a_1 = 2; \quad a_{n+1} = 3a_n$ .
$1, 2, 6, 24, 120, \cdots, a_1 = 1; \quad a_{n+1} = (n+1)a_n$ .

Is it possible in all instances to give this dual interpretation to a sequence, that is, to specify the $n^{th}$ term on the one hand and to provide a recursion definition of the sequence on the other? It is not wise to say in an absolute manner what is possible or impossible in mathematics. But at least it can be stated that sequences which are readily representable by their $n^{th}$ term may be difficult to represent by recursion and on the contrary, sequences which can be easily represented by recursion may not have an obvious $n^{th}$ term. For example, what is the recursion relation for the sequence defined by:

$$a_n = \sqrt[n]{\frac{\log n}{\sqrt[3]{n}}} \qquad ?$$

Or on the other hand, if $a_1 = 2$, $a_2 = 3$, $a_3 = 5$, and

$$a_{n+1} = \frac{7a_n + 5a_{n-1}}{a_{n-2}}$$

what is the expression for the $n^{th}$ term?

However, in most of the usual cases, it is possible to have both the $n^{th}$ term and the recursion formulation of a sequence. Many of the common sequences, for example, have their $n^{th}$ term expressed as a polynomial in n. In such a case, it is possible to find a corresponding recursion relation. In fact, for all polynomials of a given degree, there is just one recursion relation corresponding to them, apart from the initial values that are given. Let us examine this important case.

Our discussion will be based on what are known as finite differences. Given a polynomial in n, such as $f(n) = n^2 + 3n - 1$, we define

$$\Delta f(n) = f(n+1) - f(n)$$

(Read "the first difference of f(n)" for $\Delta f(n)$.). Let us carry out this operation.

$$\Delta f(n) = (n+1)^2 + 3(n+1) - 1 - (n^2 + 3n - 1)$$

$$\Delta f(n) = 2n + 4 .$$

Note that the degree of $\Delta f(n)$ is one less than the degree of the original polynomial. If we take the difference of $\Delta f(n)$ we obtain the second difference of $f(n)$. Thus

$$\Delta^2 f(n) = 2(n+1) + 4 - (2n+4) = 2$$

Finally, the third difference of $f(n)$ is $\Delta^3 f(n) = 2 - 2 = 0$. The situation portrayed here is general. A polynomial of degree $m$ has a first difference of degree $m - 1$, a second difference of degree $m - 2, \cdots$, an $m^{th}$ difference which is constant and an $(m+1)^{st}$ difference which is zero. Basically, this result depends on the lead term of highest degree. We need only consider then what happens to $f(n) = n^m$ when we take a first difference.

$$\Delta f(n) = (n+1)^m - n^m = n^m + mn^{m-1} \cdots - n^m$$

$\Delta f(n) = mn^{m-1} + \cdots$ terms of lower degree. Thus the degree drops by 1.

Suppose we designate the terms of our sequence as $T_n$. Then

$$\Delta T_n = T_{n+1} - T_n$$
$$\Delta^2 T_n = T_{n+2} - T_{n+1} - (T_{n+1} - T_n) = T_{n+2} - 2T_{n+1} + T_n$$
$$\Delta^3 T_n = T_{n+3} - 2T_{n+2} + T_{n+1} - (T_{n+2} - 2T_{n+1} + T_n)$$

or

$$\Delta^3 T_n = T_{n+3} - 3T_{n+2} + 3T_{n+1} - T_n .$$

Clearly the coefficients of the Pascal triangle with alternating signs are being generated and it is clear from the operation that this will continue.

We are now ready to transform a sequence with a term expressed as a polynomial in $n$ into a recursion relation. Consider again:

$$T_n = n^2 + 3n - 1 .$$

Take the third difference of both sides. Then

$$\Delta^3 T_n = \Delta^3(n^2 + 3n - 1)$$

But the third difference of a polynomial of the second degree is zero. Hence

$$T_{n+3} - 3T_{n+2} + 3T_{n+1} - T_n = 0$$

or

$$T_{n+3} = 3T_{n+2} - 3T_{n+1} + T_n$$

is the required recursion relation for all sequences whose term can be expressed as a polynomial of the second degree in $n$.

An interesting particular case is the arithmetic progression whose $n^{th}$ term is

$$T_n = a + (n - 1)d ,$$

where $a$ is the first term and $d$ the common difference. For example, if $a$ is 5 and $d$ is 4,

$$T_n = 5 + 4(n - 1) = 4n - 1 .$$

In any event, an arithmetic progression has a term which can be expressed as a polynomial of the first degree in $n$. Accordingly the recursion relation for all arithmetic progression is:

$$\Delta^2 T_n = 0$$

or

$$T_{n+2} = 2T_{n+1} - T_n .$$

The recursion relation for the geometric progression with ratio $r$ is evidently

$$T_{n+1} = rT_n .$$

For example, 2, 18, 54, 162, $\cdots$ is specified by $a_1 = 2$, $T_{n+1} = 3T_n$.

This takes care of our listed sequences except the factorial and the reciprocal of n. For the factorial:

$$T_{n+1} = (n + 1)T_n .$$

However, we do not have a pure recursion relation to a subsequent from previous terms of the sequence. We need to eliminate n in the coefficient to bring this about. Now

$$n = T_n / T_{n-1}$$

and

$$n + 1 = T_{n+1} / T_n .$$

Thus

$$T_{n+1} / T_n - T_n / T_{n-1} = 1$$

so that

$$T_{n+1} = T_n (T_n + T_{n-1}) / T_{n-1} .$$

Again for $T_n = 1/n$, we have

$$n = 1/T_n , \quad n + 1 = 1/T_{n+1}, \quad 1/T_{n+1} - 1/T_n = 1$$

so that

$$T_{n+1} = T_n / (1 + T_n) .$$

## RECURRING SEQUENCES — LESSON 1

## PROBLEMS

1. Find the $n^{th}$ term and the recursion relation for the sequence: 2, 6, 12, 20, 30, 42, 56, $\cdots$ .
2. Find the $n^{th}$ term and the recursion relation for the sequence: 1, 4, 7, 10, 13, 16, $\cdots$ .
3. Determine the $n^{th}$ term and the recursion relation for the sequence: 1, 8, 27, 64, 125, 216, 343, $\cdots$ .
4. For $T_1 = 1$, $T_2 = 3$ and $T_{n+1} = T_n / T_{n-1}$, find a form of expression for the $n^{th}$ term. (It may be more convenient to do this using a number of formulas.)
5. Find the recursion relation for the sequence with the term $T_n = \sqrt{n}$.
6. What is the recursion relation for a sequence whose term is a cubic polynomial in $n$?
7. If $a$ is a positive constant, determine the recursion relation for the sequence with the term $T_n = a^n$.
8. Find a recursion relation corresponding to $T_{n+1} = T_n + 2n + 1$ which does not involve $n$ except in the subscripts nor a constant except as a coefficient.
9. Find an expression(s) for the $n^{th}$ term of the sequence the recursion relation $T_n T_{n+1} = 1$, where $T_1 = a$ ($a$ not zero).
10. For the sequence with term $T_n = n/(n + 1)$, find a recursion relation with $n$ occurring only in subscripts.

(See page 55 for answers to problems.)

# LINEAR RECURSION RELATIONS
## LESSON TWO

Recursion relations can be set up at will. There is, however, a particular type known as the linear recursion relation which by its simplicity, range of application, and interesting mathematical properties deserves special consideration. In this lesson, the linear recursion relation will be described and the method of expressing its terms by means of the roots of an auxiliary equation analyzed. These basic ideas will be applied and amplified in greater detail in succeeding articles.

The term "linear" in mathematics is used by way of analogy with the equation of a straight line in the plane where the variables x and y do not enter in a degree higher than the first. By extension, there are linear equations in more variables which characterize the plane in three-space, the hyperplane in four-space, etc. By further analogy, one speaks of linear differential equations in which the dependent variable and its derivatives are not found in a degree higher than one. In this context it is natural to call a recursion relation of the form:

$$(1) \qquad T_{n+1} = a_1 T_n + a_2 T_{n-1} + a_3 T_{n-2} + \cdots + a_r T_{n-r+1}$$

where the $a_i$ are constants, a linear recursion relation. If $a_r$ is the last non-zero coefficient, then this would be spoken of as a linear recursion relation of order r.

Note that there is no allowance for a constant term. This, however, is no restriction. If, for example,

$$T_{n+1} = 3T_n - 2T_{n-1} + 4T_{n-2} + 8$$

then since

$$T_n = 3T_{n-1} - 2T_{n-2} + 4T_{n-3} + 8$$

it follows by subtraction that

$$T_{n+1} = 4T_n - 5T_{n-1} + 6T_{n-2} - 4T_{n-3} \quad ,$$

so that a linear recursion relation of the standard form (1) can be obtained from this variant.

## LINEAR RECURSION RELATION OF THE FIRST ORDER

The linear recursion relation of the first order is

(2) $$T_{n+1} = r\,T_n \quad ,$$

in which each term is a fixed multiple of the previous term. Evidently, this is the recursion relation of a geometric progression. In terms of the technique that is being developed for relating the terms of the sequence with the roots of an auxiliary equation, we set up the equation corresponding to this recursion relation, namely:

(3) $$x - r = 0 \quad ,$$

which has the one root $r$. The term of the sequence can be written as a multiple of the $n^{th}$ power of this root, thus:

$$T_n = (a/r)\,r^n \quad .$$

That this term satisfies the recursion relation (2) follows from (3), since on substituting $r$ for $x$, we have:

$$r = r \quad ,$$

and on multiplying both sides by $r^{n-1}$,

$$r^n = r \cdot r^{n-1} \, .$$

Note that the powers of the root have the same recursion relation as the terms (2)! So if

$$T_{n+1} = (a/r)\, r^{n+1}$$

and

$$r^{n+1} = r \cdot r^{n} ,$$

$$T_{n+1} = r(a/r)\, r^{n} = r\, T_{n} .$$

Perhaps due to the simplicity of this case, the considerations are confusing! But let us pass on to a second-order linear relation where the operations are not so obvious.

## SECOND-ORDER LINEAR RECURSION RELATIONS

In a subsequent article, these relations will be taken up in all possible detail to cover the various situations that may arise. But here we shall start with a simple example to show how the method operates.

Consider then a linear recursion relation

(4) $$T_{n+1} = 5\,T_{n} - 6\,T_{n-1} .$$

If all terms are brought to one side and equated to zero, the result is:

(5) $$T_{n+1} - 5\,T_{n} + 6\,T_{n-1} = 0 .$$

If now the successive terms are replaced by powers of x one obtains the auxiliary equation

(6) $$x^2 - 5x + 6 = 0$$

whose roots are $r = 3$, $s = 2$. Since they satisfy the equation (5), it follows that

$$r^2 = 5r - 6$$
$$s^2 = 5s - 6$$

Since we may multiply by any power of r or s,

(7) $$r^{n+1} = 5\,r^n - 6\,r^{n-1}$$
$$s^{n+1} = 5\,s^n - 6\,s^{n-1}$$

Note that the powers of r and s satisfy the same recursion relation (4) as the terms of the sequence $T_n$. Hence if we express these terms as linear combinations of powers of r and s, we should obtain expressions that satisfy the recursion relation (4). Set

(8) $$T_{n-1} = a\,r^{n-1} + b\,s^{n-1}$$
$$T_n = a\,r^n + b\,s^n$$

where a and b are constants. Then

$$T_{n+1} = 5\,T_n - 6\,T_{n-1} = a(5r^n - 6r^{n-1}) + b(5s^n - 6s^{n-1})$$

or

$$T_{n+1} = ar^{n+1} + bs^{n+1}$$

so that the form of the term persists for all values of n once it is established for two initial values.

What this implies is that given any two starting values $T_1 = p$, $T_2 = q$ it is possible to find a sequence

$$T_n = a3^n + b\,2^n$$

satisfying the recursion relation (4). Consider the particular case $p = 2$ $q = 7$. Then we should have:

$$2 = a \cdot 2 + b \cdot 3$$
$$7 = a \cdot 2^2 + b \cdot 3^2$$

Solving for a and b we obtain $a = -1/2$, $b = 1$, so that in general,

$$T_n = (-1/2)2^n + 3^n .$$

If the roots r and s are real and distinct with $rs \neq 0$, it will always be possible to solve the above set of equations for the determinant of the coefficients of the equations:

$$p = ar + bs$$
$$a = ar^2 + bs^2$$

is

$$\begin{vmatrix} r & s \\ r^2 & s^2 \end{vmatrix} = rs(s - r)$$

which is not zero if $rs \neq 0$ and $s \neq r$.

These considerations can be extended to relations of higher order. For example, suppose we wish to express the terms of a sequence beginning with 3, 8, 14 in the form:

$$T_n = a2^n + b3^n + c5^n .$$

It is simply necessary to set up a recursion relation with roots 2, 3, and 5. Thus the auxiliary equation would be

$$(x - 2)(x - 3)(x - 5) = 0$$

or

$$x^3 = 10x^2 - 31x + 30$$

so that

$$T_{n+1} = 10T_n - 31T_{n-1} + 30T_{n-2}$$

giving sequence terms as follows:

$$3,\ 8,\ 14,\ -18,\ -374,\ -2762,\ -16566,\cdots$$

To express $T_n$ in terms of the powers of the roots use the initial values to form equations as follows.

$$3 = 2a + 3b + 5c$$

$$8 = 4a + 9b + 25c$$

$$14 = 8a + 27b + 125c$$

from which $a = -5/6$, $b = 2$, $c = -4/15$. Thus

$$T_n = (-5/6)2^n + 2\cdot 3^n + (-4/15)5^n .$$

Evidently, there are many questions that require further study; the case of equal roots of the auxiliary equation; what happens if the roots are irrational; the situation in which the roots are complex; and various combinations of these cases. Such matters will receive attention in a number of subsequent lessons.

PROBLEMS

1. Find the recursion relation for the sequence beginning 3, 10 with terms in the form

$$T_n = a + 2^n b ,$$

and calculate the first ten terms of the sequence.

2. Given the sequence beginning with 5, 12 having a recursion relation

$$T_{n+1} = 8T_n - 15T_{n-1} ,$$

express $T_n$ as a linear combination of powers of the roots of the auxiliary equation.

3. The sequence

$$5, 13, 61, 349, 2077, 12445, 74653, 447901, \cdots$$

obeys a linear recursion relation of the second order. Find this relation and express $T_n$ as a linear combination of powers of the roots of the auxiliary equation.

4. A sequence with initial terms 3, 7, 13 has an auxiliary equation

$$x^3 - 6x^2 + 11x - 6 = 0 .$$

Express the term $T_n$ as a linear combination of powers of the roots of this equation.

5. A third-order recursion relation governs the terms of the sequence: 1, 6, 14, 45, 131, 396, 1184, 3555, 10661, 31986, 95954, 287865, 863589 . Determine the coefficients in this recursion relation and express the term $T_n$ as a linear combination of powers of the roots of the auxiliary equation.

6. A sequence has $T_1 = 1$, $T_2 = 2$ and each term thereafter determined as the arithmetic mean of the two previous terms. Find an explicit expression for $T_n$.

7. If $T_1 = 1$, $T_2 = 3$ and $T_{n+1} = (2/3)T_n + (1/3)T_{n-1}$, what is an explicit formula for the $n^{th}$ term of the sequence?

8. If $T_1 = 3$, $T_2 = 5$, and $T_{n+1} = 2T_{n-1}$, find an explicit formula for the $n^{th}$ term of the sequence.

9. For the sequence 2, 4, 2, 4, 2, 4, 2, 4, $\cdots$ what is the linear recursion relation and what is the formula for the terms of the sequence using the roots of the auxiliary equation?

10. For the sequence 1, -1, 1/2, -1/2, 1/4, -1/4, 1/8, -1/8, $\cdots$ find the linear recursion relation and determine the formula for the $n^{th}$ term of the sequence in terms of the roots of the auxiliary equation.

(See page 55 for answers to problems.)

# LINEAR RECURSION RELATIONS
# LESSON THREE—THE BINET FORMULAS

In the previous lesson, the technique of relating the terms of a linear recursion relation to the roots of an auxiliary equation was studied and illustrated. The Fibonacci sequences are characterized by the recursion relation:

(1) $$T_{n+1} = T_n + T_{n-1} ,$$

which is a linear recursion relation of the second order having an auxiliary equation:

(2) $$x^2 = x + 1$$

or

(3) $$x^2 - x - 1 = 0 .$$

The roots of this equation are:

(4) $$r = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad s = \frac{1-\sqrt{5}}{2}$$

From the theory of the relation of roots to coefficients or by direct calculation it can be ascertained that:

(5) $$r + s = 1 \quad \text{and} \quad rs = -1 .$$

It follows from what has been developed in the previous lesson that the terms of any Fibonacci sequence can be written in the form:

(5) $$T_n = ar^n + bs^n ,$$

where a and b are suitable constants. For example, let

$$T_1 = 2, \; T_2 = 5 \; .$$

The relations that must be satisfied are:

$$2 = ar + bs$$

$$5 = ar^2 + bs^2 \; .$$

These give solutions:

$$a = \frac{15 + \sqrt{5}}{10} \quad \text{and} \quad b = \frac{15 - \sqrt{5}}{10} \; ,$$

so that

$$T_n = \frac{15 + \sqrt{5}}{10} r^n + \frac{15 - \sqrt{5}}{10} s^n \; .$$

Let us apply this technique to what is commonly known as the Fibonacci sequence whose initial terms are $F_1 = 1$ and $F_2 = 1$. Then

$$1 = ar + bs$$

$$1 = ar^2 + bs^2 \; ,$$

with solutions

$$a = \frac{1}{\sqrt{5}}$$

and

$$b = \frac{-1}{\sqrt{5}}$$

so that

(6)
$$F_n = \frac{r^n - s^n}{\sqrt{5}} ,$$

the BINET FORMULA for the Fibonacci sequence.

Similarly, for the Lucas sequence with $L_1 = 1$ and $L_2 = 3$ ,

$$1 = ar + bs$$

$$3 = ar^2 + bs^2 ,$$

one obtains $a = 1$, $b = 1$, so that:

(7)
$$L_n = r^n + s^n ,$$

the BINET FORMULA for the Lucas sequence.

## THE GOLDEN SECTION RATIO

With this formulation it is easy to see the connection between the Fibonacci sequences and the Golden Section Ratio. To divide a line segment in what is known as "extreme and mean ratio" or to make a Golden Section of the line segment, one finds a point on the line such that the length of the entire line is to the larger segment as the larger segment is to the smaller segment. To produce an exact parallel with the Fibonacci sequence auxiliary equation, let $x$ be the length of the line and $1$ the length of the larger segment. Then:

$$x : 1 = 1 : 1 - x ,$$

which leads to the equation

$$x^2 - x - 1 = 0 .$$

Clearly, we are interested in the positive root

$$r = \frac{1 + \sqrt{5}}{2}$$

The other root $s = -1/r$ is the negative reciprocal of $r$, the Golden Section Ratio. (It may be noted that

$$\frac{1}{r} = \frac{\sqrt{5} - 1}{2}$$

is also considered the Golden Section Ratio by some authors. This is a matter of point of view: whether one is taking the ratio of the larger segment to the smaller segment or vice-versa.)

## USING THE BINET FORMULAS

The Binet formulas for the Fibonacci and Lucas sequences are certainly not the practical means of calculating the terms of these sequences. Algebraically, however, they provide a powerful tool for creating or verifying Fibonacci-Lucas relations. Let us consider a few examples.

Example 1

If we study the terms of the Fibonacci sequence and the Lucas sequence in the following table:

| n | $F_n$ | $L_n$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 1 | 3 |
| 3 | 2 | 4 |
| 4 | 3 | 7 |
| 5 | 5 | 11 |
| 6 | 8 | 18 |
| 7 | 13 | 29 |
| 8 | 21 | 47 |
| 9 | 34 | 76 |
| 10 | 55 | 123 |

it is a matter of observation that:

$$F_4 L_4 = 3 \times 7 = 21 = F_8$$

$$F_5 L_5 = 5 \times 11 = 55 = F_{10}$$

and in general it appears that:

$$F_n L_n = F_{2n} .$$

Why is this so? Using the Binet formula for $F_{2n}$,

$$F_{2n} = \frac{r^{2n} - s^{2n}}{\sqrt{5}} = \frac{(r^n - s^n)}{\sqrt{5}} (r^n + s^n) = F_n L_n$$

<u>Example 2</u>

$$F_{kn} = \frac{r^{kn} - s^{kn}}{\sqrt{5}} = \frac{(r^k)^n - (s^k)^n}{\sqrt{5}}$$

has a factor

$$\frac{r^k - s^k}{\sqrt{5}} = F_k ,$$

which proves that if $k$ is a divisor of the subscript of a Fibonacci number $F_m$, then $F_k$ divides $F_m$.

<u>Example 3</u>

By taking successive values of $k$, one can intuitively surmise the formula:

$$F_{n+k} F_{n-k} - F_n^2 = (-1)^{n+k+1} F_k^2$$

To prove this relation, use the Binet formula for $F$. This gives:

$$F_{n+k}F_{n-k} - F_n^2 = \frac{r^{n+k} - s^{n+k}}{\sqrt{5}} \cdot \frac{r^{n-k} - s^{n-k}}{\sqrt{5}} - \frac{(r^n - s^n)^2}{5}$$

$$= \frac{r^{2n} + s^{2n} - r^{n+k}s^{n-k} - r^{n-k}s^{n+k} - r^{2n} + 2r^n s^n - s^{2n}}{5}$$

$$= - \frac{r^{n-k}s^{n-k}(r^{2k} - 2r^k s^k + s^{2k})}{5} = (-1)^{n+k+1} F_k^2 .$$

## LINEAR RECURSION RELATIONS

## PROBLEMS

1. Prove that

$$L_{2n} = L_n^2 + 2(-1)^{n+1} .$$

2. Using the Binet formulas, find the value of:

$$L_n F_{n-1} - F_n L_{n-1} .$$

3. $F_{3n} = F_n(\qquad)$. Determine the expression for the cofactor of $F_n$.
4. $F_{5n} = F_n(\qquad)$. Determine the expression for the cofactor of $F_n$.
5. $L_{3n} = L_n(\qquad)$. Find the expression for the cofactor of $L_n$.
6. $L_{5n} = L_n(\qquad)$. Find the expression for the cofactor of $L_n$.
7. For the Fibonacci relation with $T_1 = 3$, $T_2 = 7$, find the expression for $T_n$ in terms of powers of $r$ and $s$.
8. Using the binomial expansion, find an expression for $F_n$ in terms of powers of 5 and binomial coefficients.
9. Do likewise for $L_n$.
10. Assuming the relation

$$L_n + L_{n+2} = 5F_{n+1} ,$$

determine an equivalent single Fibonacci number for $F_n^2 + F_{n+1}^2$ using the Binet formula.

(See page 56 for answers to problems.)

# LINEAR RECURSION RELATIONS – LESSON FOUR
# SECOND-ORDER LINEAR RECURSION RELATIONS

Given a second-order linear recursion relation

$$T_{n+1} = a\,T_n + b\,T_{n-1}\,, \tag{1}$$

where a and b are real numbers and the values $T_i$ of the sequence are real as well, there is an auxiliary equation:

$$x^2 - ax - b = 0\,, \tag{2}$$

with roots

$$r = \frac{a + \sqrt{a^2 + 4b}}{2}$$
$$s = \frac{a - \sqrt{a^2 + 4b}}{2}\,. \tag{3}$$

As is usual with quadratic equations, three cases may arise depending on whether

(4)

$a^2 + 4b > 0$, roots real and distinct;

$a^2 + 4b = 0$, roots real and equal;

$a^2 + 4b < 0$, roots complex numbers.

CASE 1. $a^2 + 4b > 0$.

In previous lessons we have considered cases of this kind. It has been noted that the roots may be rational or irrational. There seems to be nothing to add for the moment to the discussion of these cases.

CASE 2. $a^2 + 4b = 0$.

The presence of multiple roots in the auxiliary equation clearly requires some modification in the previous development. If

$$x^2 - ax - b = 0$$

$$x^n - ax^{n-1} - bx^{n-2} = 0 \; .$$

Since the equation has a multiple root $(a/2)$, the derivative of this equation will have this same root. Hence

$$nx^{n-1} - a(n-1)x^{n-2} - b(n-2)x^{n-3} = 0 \tag{5}$$

is satisfied by the multiple root also.

Thus the multiple root, $r$, satisfies the following two relations:

$$\begin{aligned} r^n &= ar^{n-1} + br^{n-2} \\ nr^n &= a(n-1)r^{n-1} + b(n-2)r^{n-2} \; . \end{aligned} \tag{6}$$

The result is that if we formulate $T_n$ as

$$\begin{aligned} T_n &= A \quad {}^{n} + B r^n \\ T_{n+1} &= A(n+1)r^{n+1} + Br^{n+1} \end{aligned} \tag{7}$$

it follows that

$$\begin{aligned} T_{n+2} &= a\,T_{n+1} + b\,T_n \\ &= A\left[a(n+1)r^{n+1} + bn\,r^n\right] + B\left[ar^{n+1} + br^n\right] \\ &= A(n+2)r^{n+2} + B\,r^{n+2} \end{aligned} \tag{8}$$

so that the form of T is maintained.

EXAMPLE

Find the expression for $T_n$ in terms of the roots of the auxiliary equation corresponding to the linear recursion relation

$$T_{n+1} = 6T_n - 9T_{n-1}$$

if $T_1 = 4$, $T_2 = 7$. Here the auxiliary equation is $x^2 - 6x + 9 = 0$ with a double root of 3. Hence $T_n$ has the form

$$T_n = Anx3^n + Bx3^n .$$

Using the values of $T_1$ and $T_2$

$$4 = Ax3 + Bx3$$

$$7 = 2Ax3^2 + Bx3^2$$

with solutions $A = -5/9$, $B = 17/9$. Hence

$$T_n = \frac{-5nx3^n + 17x3^n}{9} = 3^{n-2}[(-5n + 17)].$$

It may be noted that for any non-zero multiple root r, the determinant of the coefficients in the set of equations for $T_1$ and $T_2$ is

$$\begin{vmatrix} r & r \\ 2r^2 & r^2 \end{vmatrix} = -r^3$$

which is not zero, so that these equations will always have a solution.

CASE 3. $a^2 + 4b < 0$.

The case of complex roots is quite similar to that of real and distinct roots as far as determining coefficients from initial value equations is concerned. However, since we have specified that the terms of the sequence and the coefficients in the recursion relation are real, there will have to be a special relation between A and B in the expression for $T_n$:

$$T_n = A r^n + B s^n .$$

Note that r and s are complex conjugates, so that $r^n$ and $s^n$ are of the form $P + Qi$ and $P - Qi$ respectively, where P and Q are real. If $T_n$ is to be real, then A and B must be complex conjugates as well.

EXAMPLE

Find the expression for $T_n$ in terms of the roots of the auxiliary equation for the linear recursion relation

$$T_{n+1} = 3 T_n - 4 T_{n-1} ,$$

with $T_1 = 5$, $T_2 = 9$. Here the auxiliary equation is:

$$x^2 - 3x + 4 = 0$$

with roots

$$r = \frac{3 + i\sqrt{7}}{2} , \quad s = \frac{3 - i\sqrt{7}}{2} .$$

Then

$$5 = Ar + Bs$$

$$9 = A r^2 + B s^2$$

from which one finds that

$$A = \frac{21 - 11i\sqrt{7}}{28} , \quad B = \frac{21 + 11i\sqrt{7}}{28} .$$

Accordingly,

$$T_n = \left(\frac{21 - 11i\sqrt{7}}{28}\right) r^n + \left(\frac{21 + 11i\sqrt{7}}{28}\right) s^n .$$

AN ANALOGUE

Because of the similarities among second-order linear recursion relations it is possible to find close analogues among them to the Fibonacci and Lucas sequences. Let us consider as an example the second-order linear recursion relation

$$T_{n+1} = 3T_n + T_{n-1} .$$

The auxiliary equation is

$$x^2 - 3x - 1 = 0$$

with roots

$$r = \frac{1 + \sqrt{13}}{2}, \quad s = \frac{1 - \sqrt{13}}{2} .$$

If the initial terms are taken as $T_0 = 0$, $T_1 = 1$, $T_2 = 3$, then

$$1 = Ar + Bs$$

$$3 = Ar^2 + Bs^2 ,$$

with resulting values $A = 1/\sqrt{13}$ and $B = -1/\sqrt{13}$ so that

$$T_n = \frac{r^n - s^n}{\sqrt{13}} = \frac{r^n - s^n}{r - s}$$

has precisely the same form as the expression for $F_n$ with 13 replacing 5 under the square root sign.

If the relation $V_n = T_{n+1} + T_{n-1}$ is used to define the corresponding "Lucas" sequence, the terms of this sequence are:

$$V_0 = 2, \quad V_1 = 3, \quad V_2 = 11, \quad V_3 = 36, \cdots .$$

Solving for A and B from

$$3 = Ar + Bs$$

$$11 = Ar^2 + Bs^2$$

gives values of $A = 1$, $B = 1$, so that

$$V_n = r^n + s^n$$

in perfect correspondence to the expression for the Lucas sequence. As a result of this similarity, many relations in the Fibonacci-Lucas complex can be taken over (sometimes with the slight modification of replacing 5 by 13) to this pair of sequences. Thus:

$$T_{2n} = T_n V_n$$

$$T_{2n+1} = T_n^2 + T_{n+1}^2$$

$$T_{n+1}T_{n-1} - T_n^2 = (-1)^{n-1}$$

$$V_{2n} = V_n^2 + 2(-1)^{n+1}$$

$$V_n + V_{n+2} = 13T_{n+1}$$

$$V_n^2 + V_{n+1}^2 = 13(T_n^2 + T_{n+1}^2) .$$

PROBLEMS

1. For the sequence $T_1 = 1$, $T_2 = 3$, obeying the linear recursion relation

$$T_{n+1} = 3T_n + T_{n-1}$$

show that every integer divides an infinity of members of the sequence.

2. For the corresponding "Lucas" sequence, prove that if m divides n, where n is odd, then $V_m$ divides $V_n$.

3. Find the expression for the sequence $T_1 = 2$, $T_2 = 5$ in terms of the roots of the auxiliary equation corresponding to the linear recursion relation

$$T_{n+1} = 4T_n + 4T_{n-1}.$$

4. Prove that the second-order linear recursion relation

$$T_{n+1} = 2T_n - T_{n-1}$$

defines an arithmetic progression.

5. If $T_1 = a$, $T_2 = b$, find the expression for $T_n$ in terms of the roots of the auxiliary equation corresponding to $T_{n+1} = 4T_n - 4T_{n-1}$.

6. If $T_1 = i$, $T_2 = 1$ and $T_{n+1} = -T_{n-1}$, find the general expression for $T_n$ in terms of the roots of the auxiliary equation.

7. $T_1 = 3$, $T_2 = 7$, $T_3 = 17$, $T_4 = 43$, $T_5 = 113, \cdots$ are terms of a second-order linear recursion relation. Find this relation and express $T_n$ in terms of the roots of the auxiliary equation.

8. For the second-order linear recursion relation $T_{n+1} = 5T_n + T_{n-1}$ find the particular sequences analogous to the Fibonacci and Lucas sequences and express their terms as functions of the roots of the auxiliary equation.

9. For $T_1 = 5$, $T_2 = 9$, $T_{n+1} = 3T_n - 5T_{n-1}$, find $T_n$ in terms of the roots of the auxiliary equation.

10. If

$$T_n = \left(\frac{-66 + 13\sqrt{33}}{33}\right)\left(\frac{5 + \sqrt{33}}{2}\right)^n + \left(\frac{-66 - 13\sqrt{33}}{33}\right)\left(\frac{5 - \sqrt{33}}{2}\right)^n$$

determine the recursion relation obeyed by $T_n$ and find $T_1$ and $T_2$.

(See page 56 for solutions to problems.)

## LINEAR RECURSION RELATIONS — LESSON FIVE
## RECURSION RELATIONS OF HIGHER ORDER

The considerations applied to linear recursion relations of the second order form a pattern for dealing with relations of higher order. Given a linear recursion relation of the $k^{th}$ order:

(1) $$T_{n+1} = a_1T_n + a_2T_{n-1} + \cdots + a_k T_{n-k+1} ,$$

where the quantities $a_i$ and $T_i$ are real, there would be an auxiliary equation

(2) $$x^k - a_1x^{k-1} - a_2 x^{k-2} \cdots - a_k = 0 ,$$

for which there could be real and distinct roots, multiple real roots or complex roots conjugate in pairs. The major difficulty that arises in a relation of this type is the problem of determining the roots which ordinarily would be approximate in value.

As an example, consider one extension of the Fibonacci sequences, namely, adding the last three terms, or adding the last four terms, and so on. The recursion relations and corresponding auxiliary equations would be:

(3) $$T_{n+1} = T_n + T_{n-1} + T_{n-2} \quad \text{and} \quad x^3 - x^2 - 1 = 0$$

(4) $$T_{n+1} = T_n + T_{n-1} + T_{n-2} + T_{n-3} \quad \text{and} \quad x^4 - x^3 - x^2 - x - 1 = 0 .$$

If we look at the general type of this equation:

(5) $$x^k = x^{k-1} + x^{k-2} + x^{k-3} + \cdots + x + 1,$$

it appears that since

(6) $$2^k - 1 = 2^{k-1} + 2^{k-2} + 2^{k-3} + \cdots + 2 + 1 ,$$

there should be a root near 2. The following table gives an approximation to this root for various values of k.

| k | Approximation to Root near 2 |
|---|---|
| 3 | 1.83928676 |
| 4 | 1.92756198 |
| 5 | 1.96594824 |
| 6 | 1.98358285 |
| 7 | 1.99196420 |
| 8 | 1.99603118 |
| 9 | 1.99802948 |

Approximations, such as these, to real or complex roots can be determined, but expressing $T_n$ in terms of them does not seem very satisfying. Nevertheless, as will be seen in a subsequent lesson, such evaluations of roots of the auxiliary equation provide interesting information regarding the generated sequence.

## MULTIPLE ROOTS

The case of multiple roots calls for additional consideration. If a polynomial equation

$$a_0 x^k + a_1 x^{k-1} + a_2 x^{k-2} + \dots + a_k = 0 \tag{7}$$

has a root of multiplicity s, then (7) can be written:

$$(x - r)^s F(x) = 0 , \tag{8}$$

where F(x) is a polynomial of degree k - s. Clearly, this equation and the equations formed by setting the first s - 1 derivatives equal to zero are all satisfied by r. This provides a clue for dealing with roots of any multiplicity when found in the auxiliary equation of a recursion relation. For concreteness, let us consider a root r of multiplicity 3.

Let the equation having this multiple root be:

(9) $$x^3 - ax^2 - bx - c = 0 .$$

Multiply both sides of the equation by $x^n$ to obtain:

(10) $$x^{n+3} - ax^{n+2} - bx^{n+1} - cx^n = 0 .$$

Take the derivative and set the resulting polynomial equal to zero.

(11) $$(n+3)x^{n+2} - a(n+2)x^{n+1} - b(n+1)x^n - cnx^{n-1} = 0 .$$

Repeat this operation on (11).

(12) $$(n+3)(n+2)x^{n+1} - a(n+2)(n+1)x^n - b(n+1)nx^{n-1} - cn(n-1)x^{n-2} = 0.$$

The multiple root $r$ must satisfy the relations (10), (11), and (12) so that on replacing $x$ by $r$ and multiplying (11) by $r$ and (12) by $r^2$ we have the following three recursion relations for $r$.

(13) $$r^{n+3} = ar^{n+2} + br^{n+1} + cr^n ,$$

(14) $$(n+3)r^{n+3} = a(n+2)r^{n+2} + b(n+1)r^{n+1} + cnr^n ,$$

(15) $$(n+3)(n+2)r^{n+3} = a(n+2)(n+1)r^{n+2} + b(n+1)nr^{n+1} + cn(n-1)r^n .$$

On the basis of these recursion relations the indicated expression for $T_n$ is:

(16) $$T_n = An(n-1)r^n + Bnr^n + Cr^n .$$

We show first that this relation continues to hold for succeeding values of $n$ if it is true for three consecutive values. For if

(17) $$T_{n+1} = A(n+1)nr^{n+1} + B(n+1)r^{n+1} + Cr^{n+1} ,$$

and

(18) $$T_{n+2} = A(n+2)(n+1)r^{n+2} + B(n+2)r^{n+2} + C\,r^{n+2} ,$$

then

$$T_{n+3} = a\,T_{n+2} + b\,T_{n+1} + c\,T_n$$

is equal to:

19) $$T_{n+3} = A(n+3)(n+2)r^{n+3} + B(n+3)r^{n+3} + C\,r^{n+3}$$

on the basis of relations (13), (14), and (15).

Given three initial values $T_1$, $T_2$, and $T_3$, the relations they should satisfy on the basis of (16) would be:

(20) $$\begin{aligned} T_1 &= Br + Cr \\ T_2 &= 2Ar^2 + 2Br^2 + Cr^2 \\ T_3 &= 6Ar^3 + 3Br^3 + Cr^3 \end{aligned} .$$

The determinant of the coefficients of the unknowns A, B, C has a value of $-2r^6$, so that if r is not zero, there are unique solutions for A, B, and C. Thus three initial values $T_1$, $T_2$ and $T_3$ can be expressed in the form given by (16). It follows that this form will continue to hold for all values of n.

It may be noted in passing that if the multiple root has a value of 1, $T_n$ reduces to a polynomial in n.

Example. Express the terms of the recursion relation

$$T_{n+1} = 7T_n - 17T_{n-1} + 14T_{n-2} + 4T_{n-3} - 8T_{n-4}$$

in terms of the roots of the auxiliary equation:

$$x^5 - 7x^4 + 17x^3 - 14x^2 - 4x + 8 = 0.$$

By synthetic division three equal roots, 2, are found and the residual quadratic has the roots

$$\frac{1+\sqrt{5}}{2} \quad \text{and} \quad \frac{1-\sqrt{5}}{2} \quad .$$

Accordingly,

$$T_{n+1} = An(n-1)x2^n + Bnx2^n + Cx2^n + Dr^n + Es^n$$

where

$$r = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad s = \frac{1-\sqrt{5}}{2}$$

<u>Example.</u> For the recursion relation

$$T_{n+1} = 3T_n - 3T_{n-1} + T_{n-2} \ ,$$

with initial values $T_1 = 5$, $T_2 = 8$, $T_3 = 17$, express $T_n$ in terms of the roots of the auxiliary equation.

This equation is

$$x^3 - 3x^2 + 3x - 1 = 0 \ ,$$

which has a triple root of 1. Thus

$$T_{n+1} = An(n-1) + Bn + C \ ,$$

a polynomial in n. Then

$$\begin{aligned} 5 &= B + C \\ 8 &= 2A + 2B + C \\ 17 &= 6A + 3B + C \quad , \end{aligned}$$

leading to the values $A = 3$, $B = -3$, $C = 8$, so that

$$T_{n+1} = 3n^2 - 6n + 8 .$$

PROBLEMS

1. Find the recursion relation satisfied by

$$T_n = 3n^2 - 5n + 4 + 2 \times 5^n .$$

2. Given the recursion relation

$$T_{n+1} = 6T_n - 11T_{n-1} + 6T_{n-2} ,$$

and initial values

$$T_1 = 8, \qquad T_2 = 15, \qquad T_3 = 22 .$$

Express the general term $T_n$ in terms of the roots of the auxiliary equation.

3. $S_n$ is the Fibonacci sequence 3, 7, 10, 17, 27, $\cdots$, and $R_n$ is the geometric progression 5, 15, 45, 135, $\cdots$

$$T_n = R_n + S_n .$$

Find the recursion relation for $T_n$.

4. If $T_n = 3n + 2 + 2(-1)^n + F_n$, find the recursion relation for $T_n$.

5. If $T_1 = 13$, $T_2 = 15$, $T_3 = 22$, and $T_{n+1} = 4T_n - T_{n-1} - 2T_{n-2}$, express $T_n$ in terms of the roots of the auxiliary equation of this recursion relation.

6. If $T_1 = 5$, $T_2 = 7$, $T_3 = 10$, and $T_{n+1} = 7T_{n+1} + 6T_{n-2}$, find the explicit formula for $T_n$ in terms of the roots of the auxiliary equation.

7. The sequence 1, 4, 6, 10, 10, 18, 10, 34, -6, 82, -84, $\cdots$ is a third-order linear recursion sequence. Find the expression for this linear recursion relation.

8. If $T_1 = 4$, $T_2 = 7$, $T_3 = 13$, and $T_{n+1} = 3T_{n-1} - 2T_{n-2}$, find the expression for $T_n$ in terms of the roots of the auxiliary equation.

9. Given that $T_{n+1} = T_n + 3n + 2$, find (1) the linear recursion relation for $T_i$. (2) With $T_1 = 4$, $T_2 = 9$, $T_3 = 17$, find an explicit expression for $T_n$.

10. For the sequence 1, 3, -1, -3, 1, 3, -1, -3, 1, 3, -1, -3, $\cdots$ determine the linear recursion relation and find the expression for $T_n$ in terms of the roots of the auxiliary equation.

(See page 58 for solutions to problems.)

# LINEAR RECURSION RELATIONS — LESSON SIX
## COMBINING LINEAR RECURSION RELATIONS

Suppose we have two sequences $P_i$(1, 5, 25, 125, 625, 3125, ···) with a recursion relation

(1) $$P_{n+1} = 5P_n$$

and $Q_i$(3, 10, 13, 23, 36, 59, ···), a Fibonacci sequence with recursion relation:

(2) $$Q_{n+1} = Q_n + Q_{n-1} .$$

Let

(3) $$T_n = P_n + Q_n .$$

What is the recursion relation of $T_n$ and how can it be conveniently obtained from the recursion relations of $P_n$ and $Q_n$?

Proceeding in a straightforward manner, we could first eliminate $P_n$ as follows:

$$T_{n+1} = P_{n+1} = Q_{n+1}$$

$$5T_n = 5P_n + 5Q_n .$$

Subtracting and using relation (1),

$$T_{n+1} - 5T_n = Q_{n+1} - 5Q_n .$$

We can proceed likewise for Q. Thus

$$\begin{aligned} T_{n+1} - 5T_n &= Q_{n+1} - 5Q_n \\ T_n - 5T_{n-1} &= Q_n - 5Q_{n-1} \\ T_{n-1} - 5T_{n-2} &= Q_{n-1} - 5Q_{n-2} . \end{aligned}$$

Now subtract the sum of the last two equations from the first and use relation (2). The result is:

$$T_{n+1} - 6T_n + 4T_{n-1} + 5T_{n-2} = 0 ,$$

a recursion relation involving only $T_i$.

A much simpler approach is by means of an operator E, such that

(3) $$(E)T_n = T_{n+1} .$$

The effect of E is to increase the subscript by 1. A relation

$$P_{n+1} - 5P_n = 0 ,$$

can be written

$$(E - 5)P_n = 0 ,$$

and a relation

$$Q_{n+1} - Q_n - Q_{n-1} = 0 ,$$

can be written

$$(E^2 - E - 1)Q_{n-1} = 0 .$$

It is not difficult to convince onself that these operators obey the usual algebraic laws. As a result, if

$$T_n = P_n + Q_n ,$$

$(E - 5)(E^2 - E - 1)T_n = (E - 5)(E^2 - E - 1)P_n + (E - 5)(E^2 - E - 1)Q_n$. But $(E - 5)P_n = 0$ and $(E^2 - E - 1)Q_n = 0$, so that

$$(E - 5)(E^2 - E - 1)T_n = 0$$

or

$$(E^3 - 6E^2 + 4E + 5)T_n = 0 ,$$

which is equivalent to the recursion relation

$$T_{n+3} = 6T_{n+2} - 4T_{n+1} - 5T_n .$$

In general, if we have linear operators such that:

$$f(E)P_n = 0 \quad \text{and} \quad g(E)Q_n = 0 \quad \text{and} \quad T_n = AP_n + BQ_n,$$

where A and B are constants, then

$$f(E)g(E)T_n = Af(E)g(E)P_n + Bf(E)g(E)Q_n = 0 ,$$

since $f(E)P_n = 0$, and $g(E)Q_n = 0$. Thus when $T_n$ is the sum of terms of two sequences with different recursion relations, the recursion relation for $T_n$ is found by multiplying $T_n$ by the two recursion operators for the two sequences.

Example. What is the recursion relation for $T_n = 2 \times 5^n + n^2 - n + 4$? The recursion relation for $2 \times 5^n$ is $(E - 5)P_n = 0$, and that for $n^2 - 4 + 4$ is $(E^3 - 3E^2 + 3E - 1)Q_n = 0$. Thus the recursion relation for the given sequence is

$$(E - 5)(E^3 - 3E^2 + 3E - 1)T_n = 0,$$

which is equivalent to:

$$T_{n+4} = 8T_{n+3} - 18T_{n+2} + 16T_{n+1} - 5T_n .$$

Example. Find the recursion relation corresponding to $T_n$ if

$P_{n+1} = P_n + P_{n-1} + P_{n-2}$ and $Q_n = 3n^2 - 4n + 5$ and $T_n = P_n + Q_n$.

The operator expressions for these recursion relations are:

$$(E^3 - E^2 - E - 1)P_{n-2} = 0 \quad \text{and} \quad (E^3 - 3E^2 + 3E - 1)Q_{n-2} = 0 .$$

Thus the recursion relation for $T_n$ is:

$$(E^3 - E^2 - E - 1)(E^3 - 3E^2 + 3E - 1)T_n = 0 ,$$

which is equivalent to

$$T_{n+6} = 4T_{n+5} - 5T_{n+4} + 2T_{n+3} - T_{n+2} + 2T_{n+1} - T_n .$$

It may be noted that two apparently different recursion relations may conceal the fact that they embody partly identical recursion relations. For example, if

$$\begin{aligned} P_n &= 4P_{n-1} - 3P_{n-2} - 2P_{n-3} + P_{n-4} \\ Q_n &= 3Q_{n-1} - 2Q_{n-2} - Q_{n-3} + Q_{n-4} \end{aligned} ,$$

and we proceed directly to find the recursion operator and corresponding recursion relation for $T_n = P_n + Q_n$, we arrive at a recursion relation of order eight. However, in factored form, we have:

$$(E^2 - E - 1)(E^2 - 3E + 1)P_{n-4} = 0 ,$$

and

$$(E^2 - E - 1)(E^2 - 2E + 1)Q_{n-4} = 0 .$$

The recursion relation for $T_n$ in simpler form would thus be:

$$(E^2 - E - 1)(E^2 - 3E + 1)(E^2 - 2E + 1)T_n = 0 ,$$

which is only of order six.

If the terms of the two sequences are given explicitly, a slightly different but equivalent procedure using the auxiliary equation is possible. Thus if

$$P_n = 5n + 2 + 2 \times 3^n + F_n$$
$$Q_n = n^2 - 3n + 5 - 6 \times 2^n + L_n ,$$

the roots of the auxiliary equation for $P_n$ are 1, 1, 3, r, and s, while those of the auxiliary equation for $Q_n$ are 1, 1, 1, 2, r, s. Hence the roots for the auxiliary equation of $T_n$ would be 1, 1, 1, 2, 3, r, s, where r and s are the roots of the equation $x^2 - x - 1 = 0$. Thus the auxiliary equation for $T_n$ would be:

$$(x - 1)^3(x - 2)(x^2 - x - 1) = 0$$

which leads equivalently to the recursion relation

$$T_{n+7} = 9T_{n+6} - 31T_{n+5} + 50T_{n+4} - 33T_{n+3} - 5T_{n+2} + 17T_{n+1} - 6T_n .$$

## PROBLEMS

1. If $P_n$ is the geometric progression 3, 15, 75, 375, 1875, $\cdots$ and

$$Q_n = 5F_n + 2(-1)^n ,$$

what is the recursion relation for $T_n = P_n + Q_n$?

2. Given recursion relations

$$P_{n+1} = 4P_n - P_{n-1} - 6P_{n-2} \qquad Q_{n+1} = 6Q_n = 10Q_{n-1} + Q_{n-2} + 6Q_{n-3},$$

with $T_{n+1} = P_{n+1} + Q_{n+1}$, determine the recursion relation of lowest order satisfied by $T_{n+1}$.

3. Determine the recursion relation for $T_n = P_n + Q_n$ where $P_n$ is the arithmetic progression 3, 7, 11, 15, 19, ⋯ and $Q_n$ is the geometric progression 2, 6, 18, 54, ⋯ .

4. Determine the recursion relation for $T_n = 2^n + F_n^2$ given that the recursion relation for $F_n^2$ is

$$F_{n+1}^2 = 2F_n^2 + 2F_{n-1}^2 - F_{n-2}^2 \, .$$

5. Determine the recursion relation for

$$T_n = 5L_n^2 + (-1)^{n-1} + 4F_n \, .$$

6. If $P_n = 3^n + 2n - 4$ and $Q_n = 2^n - 3n + 2$, find the linear recursion relation for $4P_n + 5Q_n$.

7. Given the sequences 1, -1, -2, 2, 1, -1, -2, 2, ⋯, and 1, 3, -1, -3, 1, 3, -1, -3, ⋯ find the linear recursion relation for the sum of the sequences and an explicit expression for the $n^{th}$ term in terms of the roots of the auxiliary equation.

8. If $P_n = L_n + 2n - 3$ and $Q_n = F_n + n^2$, find the linear recursion relation for the sum $P_n + Q_n$.

9. $P_n = 2 \times 3^n + 5n + 4$ and $Q_n = F_n + 2n - 3$. Find the linear recursion relation for the sum $P_n + Q_n$.

10. If $P_n = 2^n + F_n$ and $Q_n = 3^n + V_n$, determine the linear recursion relation for the sum $P_n + Q_n$.

$$V_1 = 1, \quad V_2 = 3, \quad V_{m+1} = 3V_m + V_{m-1} \, .$$

(See page 58 for solutions to problems.)

# LINEAR RECURSION RELATIONS — LESSON SEVEN
## ANALYZING LINEAR RECURSION SEQUENCES

Frequently one encounters problems such as the following: Find the next three terms in the following sequences:

$$1,\ 3,\ 5,\ 7,\ 9,\ 11,\ \cdots$$
$$3,\ 4,\ 7,\ 11,\ 18,\ 29,\ \cdots$$
$$3,\ 6,\ 12,\ 24,\ 48,\ 96,\ \cdots$$

As has been pointed out many times, the solution to such problems is highly indeterminate. It is "obvious" that the general term of the first sequence is

$$T_n = 2n - 1$$

But

$$T_n = 2n - 1 + (n - 1)(n - 2)(n - 3)(n - 4)(n - 5)(n - 6)V_n$$

where $V_n$ is the $n^{th}$ term of any sequence of finite quantities would do just as well. Similarly for the other cases.

Or looking at the matter from the standpoint of linear recursion relations, the six numbers in each case might be the first six terms of a linear recursion relation of the sixth order. Hence any infinite number of possibilities arises.

How can the problem be made more specific? Possibly, one might say: Find the expression for the $n^{th}$ term of a linear recursion relation of minimum order. Whether this is sufficient to handle all instances of this type is an open question, but it would seem to take care of the present cases.

The solutions in the three instances listed above are:

$$T_{n+1} = 2T_n - T_{n-1}$$
$$T_{n+1} = T_n + T_{n-1}$$
$$T_{n+1} = 2T_n$$

If a sequence has terms which were derived from a polynomial expression in n, this expression can be found by the method of differences. As was pointed out in the first lesson, if the terms derive from a polynomial of degree k, the $k^{th}$ differences are constant and the $(k+1)^{st}$ difference is zero. A simple method of reconstituting the polynomial is to use Newton's Interpolation Formula:

(1) $$f(n) = \frac{\Delta^k f(0)}{k!} n^{(k)} + \frac{\Delta^{k-1} f(0) n^{(k-1)}}{(k-1)!} + \cdots + \frac{\Delta f(0) n^{(1)}}{1!} + f(0)$$

where $\Delta^k f(0)$ is the $k^{th}$ difference taken at the zero value and $n^{(k)}$ is the factorial $n(n-1)(n-2)\cdots(n-k+1)$ of k terms.

<u>Example.</u> Determine the polynomial of lowest degree which fits the following set of values.

| n | f(n) | $\Delta f(n)$ | $\Delta^2 f(n)$ | $\Delta^3 f(n)$ |
|---|---|---|---|---|
| 0 | 6 | | | |
| | | 5 | | |
| 1 | 11 | | 32 | |
| | | 37 | | 18 |
| 2 | 48 | | 50 | |
| | | 87 | | 18 |
| 3 | 135 | | 68 | |
| | | 155 | | 18 |
| 4 | 290 | | 86 | |
| | | 241 | | 18 |
| 5 | 531 | | 104 | |
| | | 345 | | 18 |
| 6 | 876 | | 122 | |
| | | 467 | | 18 |
| 7 | 1343 | | 140 | |
| | | 607 | | |
| 8 | 1950 | | | |

Using Newton's Interpolation Formula,

$$f(n) = \frac{18}{3!} n(n-1)(n-2) + \frac{32}{2!} n(n-1) + 5n + 6$$

$$f(n) = 3n^3 + 7n^2 - 5n + 6.$$

Suppose that we have a sequence whose terms are the sum of the terms of two sequences: (1) A sequence whose values derive from a polynomial: (2) A sequence whose terms form a geometric progression. Is it possible to determine the components of this sequence?

Imagine that the terms of the sequence have been separated into their two component parts. Then on taking differences, the effect of the polynomial will eventually become nil. How does a geometric progression function under differencing? This can be seen from the table below.

| | | | |
|---|---|---|---|
| $a$ | | | |
| $ar$ | $a(r-1)$ | | |
| $ar^2$ | $ar(r-1)$ | $a(r-1)^2$ | |
| $ar^3$ | $ar^2(r-1)$ | $ar(r-1)^2$ | $a(r-1)^3$ |
| $ar^4$ | $ar^3(r-1)$ | $ar^2(r-1)^2$ | $ar(r-1)^3$ |
| | | $ar^3(r-1)^2$ | $ar^2(r-1)^3$ |

Clearly, differencing a geometric progression produces a geometric progression with the same ratio. By examining the form of the leading term, one can readily deduce the value of $a$, the initial term of the geometric progression as well.

<u>Example.</u>

POLYNOMIAL AND GEOMETRIC PROGRESSION COMBINED

| $n$ | $T_n$ | | | | |
|---|---|---|---|---|---|
| 1 | 4 | | | | |
| 2 | 16 | 12 | | | |
| 3 | 70 | 54 | 42 | | |
| 4 | 224 | 154 | 100 | 58 | |
| 5 | 616 | 392 | 238 | 138 | 80 |
| 6 | 1624 | 1008 | 616 | 378 | 240 |
| 7 | 4346 | 2722 | 1714 | 1098 | 720 |
| 8 | 12040 | 7644 | 4972 | 3258 | 2160 |
| 9 | 34444 | 22404 | 14710 | 9738 | 6480 |

In the last column, one has a geometric progression with ratio 3, but not in the previous column. Hence the polynomial that was combined with the geometric progression was of degree 3. For the geometric progression, $r = 3$ and

$$a \times 2^4 = 80, \text{ so that } a = 5 .$$

Eliminating the effect of the geometric progression from the leading terms gives:

$$58 - 2^3 \times 5 = 18$$
$$42 - 2^2 \times 5 = 22$$
$$12 - 2 \times 5 = 2$$
$$4 - 5 = -1$$

To apply Newton's Formula, we have to go back to the zero elements by extrapolation.

$$\Delta^3 f(0) = 18, \quad \Delta^2 f(0) = 22 - 18 = 4, \quad \Delta f(0) = 2 - 4 = -2$$
$$f(0) = -1 - (-2) = 1 .$$

Hence

$$f(n) = \frac{18}{3!} n(n - 1)(n - 2) + \frac{4}{2!} n(n - 1) - 2n + 1$$
$$f(n) = 3n^3 - 7n^2 + 2n + 1 .$$

Hence the term of the sequence has the form:

$$T_n = 3n^3 - 7n^2 + 2n + 1 + 5 \times 3^{n-1} .$$

The recursion relation for this term can be readily found by the methods of the previous lesson.

## POLYNOMIAL AND FIBONACCI SEQUENCE

If we know that the terms of a sequence are formed by combining the elements of a polynomial sequence and a Fibonacci sequence, we have a situation similar to the previous case. For whereas the polynomial element vanishes on taking a sufficient number of differences, the Fibonacci element persists. This can be seen from the following table.

| n | $T_n$ | $\Delta T_n$ | $\Delta^2 T_n$ | $\Delta^3 T_n$ |
|---|---|---|---|---|
| 1 | $T_1$ | | | |
| | | $T_0$ | | |
| 2 | $T_2$ | | | |
| | | $T_1$ | | |
| 3 | $T_3$ | | $T_{-1}$ | |
| | | $T_2$ | | $T_{-2}$ |
| 4 | $T_4$ | | $T_0$ | |
| | | $T_3$ | | $T_{-1}$ |
| 5 | $T_5$ | | $T_2$ | |
| | | $T_4$ | | $T_0$ |
| 6 | $T_6$ | | $T_3$ | |
| | | $T_5$ | | $T_1$ |
| 7 | $T_7$ | | $T_4$ | |
| | | $T_6$ | | $T_2$ |
| 8 | $T_8$ | | | |

Example.

| n | $T_n$ | | | | |
|---|---|---|---|---|---|
| 1 | 2 | | | | |
| | | 6 | | | |
| 2 | 8 | | 21 | | |
| | | 27 | | 23 | |
| 3 | 35 | | 44 | | -6 |
| | | 71 | | 17 | |
| 4 | 106 | | 61 | | 5 |
| | | 132 | | 22 | |
| 5 | 238 | | 83 | | -1 |
| | | 215 | | 21 | |
| 6 | 453 | | 104 | | 4 |
| | | 319 | | 25 | |
| 7 | 772 | | 129 | | 3 |
| | | 448 | | 28 | |
| 8 | 1220 | | 157 | | |
| | | 605 | | | |
| 9 | 1825 | | | | |

The last column has a Fibonacci property, but the previous column does not. Hence the polynomial must have been of degree three. We identify the first terms of the Fibonacci sequence as being 3, the zero term as 4, the term with -1 subscript as -1, etc. The effect of these terms can be eliminated from the leading edge of the table to give: 23 - 5 = 18; 21 - (-1) = 22; 6 - 4 = 2; 2 - 3 = -1. Calculating the zero differences as before, the final form of the term to be found is:

$$T_n = 3n^3 - 7n^2 + 2n + 1 + V_n$$

where $V_1 = 3$, $V_2 = 7$, and $V_{n+1} = V_n + V_{n-1}$.

## PROBLEMS

1. Determine the polynomial for which $f(1) = -4$; $f(2) = 22$; $f(3) = 100$; $f(4) = 200$; $f(5) = 532$; $f(6) = 946$; $f(7) = 1532$; $f(8) = 2320$.

2. The following sequence of values correspond to terms $T_1$, $T_2$, etc. of a sequence which is the sum of a polynomial and a Fibonacci sequence: 0, 4, 12, 29, 53, 87, 132, 192, 272, 391. Determine the polynomial and the Fibonacci sequence components.

3. The values: 13, 72, 227, 526, 1023, 1784, 2899, 4506, 6839 include a polynomial component and a geometric progression component. Determine the general form of the term of the sequence.

4. The sequence values: 4, 14, 12, 22, 20, 30, 28, 38, 36, ··· combine a polynomila and a geometric progression. Determine the general form of the term of the sequence.

5. The sequence values: 7, 19, 45, 109, 219, 395, 653, 1017, 1515 have a polynomial and a Fibonacci component. Determine the general form of the polynomial and find the Fibonacci sequence.

6. The following sequence has its terms the sum of corresponding terms of a geometric and arithmetic progression: 41, 73, 150, 407, 1384, 5241, 20618, 82075, 327852, 1310909, ···. Identify the component sequences.

7. The following sequence has its terms the sum of corresponding terms of an arithmetic progression and a Fibonacci sequence: 8, 17, 24, 34, 45, 60, 80, 109, 152, 218, ···. Identify the component sequences.

8. The following sequence has its terms the sum of two sequences, one geometric and the other with a general term that can be represented by a polynomial: 10, 26, 71, 201, 584, 1724, 5133, 15347, 45974, 137878, ···. Determine the two component sequences.

9. The sequence 7, 8, 15, 33, 61, 103, 162, 245, 362, 530, ··· is the sum of a Fibonacci sequence and a sequence that can be represented by a polynomial. Find the two sequences.

10. The sequence 11, 23, 40, 75, 139, 262, 497, 951, 1832, 3551, ··· is the sum of a Fibonacci sequence and a geometric progression. Determine the two component sequences.

(See page 59 for answers to problems.)

# LINEAR RECURSION RELATIONS — LESSON EIGHT
## ASYMPTOTIC RATIOS IN RECURSION RELATIONS

One of the marvels associated with Fibonacci sequences is the fact that for all such sequences the limit of the ratio $T_{n+1}/T_n$ as n approaches infinity is the Golden Section Ratio

$$\frac{1+\sqrt{5}}{2}.$$

The following table shows the ratio of successive terms for the Fibonacci sequence 2, 5, 7, 12, 19, 31, ··· .

| n | $T_n$ | $T_n/T_{n-1}$ |
|---|---|---|
| 1 | 2 | |
| 2 | 5 | |
| 3 | 7 | 1.4000000 |
| 4 | 12 | 1.7142857 |
| 5 | 19 | 1.5833333 |
| 6 | 31 | 1.6315789 |
| 7 | 50 | 1.6129032 |
| 8 | 81 | 1.6200000 |
| 9 | 131 | 1.6172839 |
| 10 | 212 | 1.6183206 |
| 11 | 343 | 1.6179245 |
| 12 | 555 | 1.6180758 |
| 13 | 898 | 1.6180180 |
| 14 | 1453 | 1.6180400 |
| 15 | 2351 | 1.6180316 |
| 16 | 3804 | 1.6180348 |
| 17 | 6155 | 1.6180336 |
| 18 | 9959 | 1.6180341 |
| 19 | 16114 | 1.6180339 |

But is this indeed so remarkable? There are many other sequences which have limiting ratios and likewise some in which there is no limit. For example, in the Tribonacci Sequence: 1, 2, 4, 7, 13, ··· where the last three terms are added together to get the next term successive ratios are as shown in the following table.

| n | $T_n$ | $T_n/T_{n-1}$ |
|---|---|---|
| 1 | 1 | |
| 2 | 2 | |
| 3 | 4 | |
| 4 | 7 | 1.7500000 |
| 5 | 13 | 1.8571428 |
| 6 | 24 | 1.8461538 |
| 7 | 44 | 1.8333333 |
| 8 | 81 | 1.8409090 |
| 9 | 149 | 1.8395061 |
| 10 | 274 | 1.8389261 |
| 11 | 504 | 1.8394160 |
| 12 | 927 | 1.8392857 |
| 13 | 1705 | 1.8392664 |

A recursion relation: $T_{n+1} = 3T_n - 4T_{n-1}$ yields a sequence which does not have a limiting ratio. For example, if $F_1 = 5$, $T_2 = 9$, the ratios are as shown in the following table.

| n | $T_n$ | $T_n/T_{n-1}$ |
|---|---|---|
| 1 | 5 | |
| 2 | 9 | |
| 3 | 7 | 0.7777777 |
| 4 | -15 | -2.1428571 |
| 5 | -73 | 4.8666666 |
| 6 | -159 | 2.1780821 |
| 7 | -185 | 1.1635220 |
| 8 | 81 | -0.4378378 |
| 9 | 983 | 12.1358024 |
| 10 | 2625 | 2.6703967 |
| 11 | 3943 | 1.5020952 |
| 12 | 1329 | 0.3370530 |
| 13 | -11785 | -8.8675696 |

Clearly, several questions emerge:

1. Which sequences have a limiting ratio?
2. Which sequences do not have a limiting ratio?
3. On what does the limiting ratio depend?

These questions can be answered conveniently on the basis of expressing $T_n$ in terms of the roots of the auxiliary equation

## THE FIBONACCI SEQUENCE

Consider the sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, $\cdots$ . Here,

$$F_n = \frac{r^n - s^n}{\sqrt{5}} ,$$

where

$$r = \frac{1 + \sqrt{5}}{2} = 1.61803 \cdots ,$$

and

$$s = \frac{1 - \sqrt{5}}{2} = -0.61803 \cdots .$$

The

$$\lim_{n \to \infty} F_n / F_{n-1} = \frac{r^n - s^n}{r^{n-1} - s^{n-1}} .$$

Dividing the terms of numerator and denominator by the $(n-1)^{st}$ power of $r$, this ratio takes the form

$$\lim_{n \to \infty} \frac{r - s(s/r)^{n-1}}{1 - (s/r)^{n-1}} .$$

Since the absolute value of $s/r$ is less than 1, the limit of the $(n-1)^{st}$ power of this ratio as $n$ goes to infinity is zero. Thus

$$\lim_{n \to \infty} F_n / F_{n-1} = r .$$

A similar analysis can be made for any Fibonacci sequence. We have found that for such a sequence,

$$T_n = Ar^n + Bs^n .$$

Therefore,

$$\lim_{n\to\infty} T_{n+1}/T_n = \frac{Ar^{n+1} + Bs^{n+1}}{Ar^n + Bs^n}$$

$$= \lim_{n\to\infty} \frac{r + (B/A)s(s/r)^n}{1 + (B/A)(s/r)^n} = r .$$

One thing we can learn from this analysis is that the root with larger absolute value, r, dominates the root with smaller absolute value, s.

## REAL AND UNEQUAL ROOTS

Clearly, if

$$T_n = Ar^n + Bs^n + Ct^n \cdots ,$$

where the roots are real and unequal and $r > s > t \cdots$ then the limiting ratio of $T_{n+1}/T_n$ will be r.

## EQUAL AND REAL ROOTS

If some of the roots are equal, but there is another real root which has the largest absolute value, this latter root will dominate to give the limiting ratio in the sequence. If the equal roots have the largest absolute value, then (consider three equal roots, r).

$$T_n = (An^2 + Bn + C)r^n + Ds^n + Et^n \cdots .$$

Therefore $\lim_{n\to\infty} T_{n+1}/T_n$ will equal

$$\lim_{n\to\infty} \frac{\{A(n + 1)^2 + B(n + 1) + C\}r^{n+1} + Ds^{n+1} + Et^{n+1} \cdots}{\{An^2 + Bn + C\}r^n + Ds^n + Et^n \cdots}$$

$$\lim_{n\to\infty} \frac{\{(n + 1)^2/n^2 + (B/A)(n + 1)/n^2 + C/(An^2)\}r + (Ds/An^2)(s/r)^n \cdots}{1 + B/(An) + C/(An^2) + (D/An^2)(s/r)^n \cdots} = r$$

Thus the dominant real root again determines the limit of the sequence ratio.

## COMPLEX ROOTS

For the type of linear recursion relation we are considering in which the coefficients are real numbers, the complex roots of the auxiliary equation occur in conjugate pairs. Let the portion of $T_n$ dependent on these roots be given by

$$cr^n + c'(r')^n ,$$

where $c$ and $c'$ are complex conjugate coefficients. Now set:

$$c = Ce^{\lambda i} \quad \text{and} \quad c' = Ce^{-\lambda i}$$
$$r = Re^{\phi i} \quad \text{and} \quad r' = Re^{-\phi i}$$

where $C$ and $R$ are the absolute values of the complex quantities $c$ and $r$, respectively. Then

$$cr^n + c'(r')^n = CR^n e^{(\lambda + n\phi)i} + CR^n e^{-(\lambda + n\phi)i}$$
$$= 2CR^n \cos(\lambda + n\phi) .$$

If there is a real root with greater absolute value than $R$, this real root will dominate and the sequence ratio will converge. However, if $R$ is greater than any of the real roots, it will dominate them. Only the cosine factor involving $n$ will not converge either directly or in ratio. Thus a sequence in which there is a pair of complex roots whose absolute value is greater than the absolute value of any of the real roots will be a sequence without a limiting ratio.

## A COROLLARY

Suppose we are seeking the roots of the cubic

$$x^3 - 7x^2 + 8x - 4 = 0 .$$

From one point of view this might be looked upon as the auxiliary equation of the recursion relation

$$T_{n+1} = 7T_n - 8T_{n-1} + 4T_{n-2}.$$

If we then calculate the terms of a sequence obeying this relation and find that their ratio approaches a limit with increasing n, this limiting ratio would correspond to the largest real root of the cubic. In the present instance, this ratio comes out to be 5.7245767.

PROBLEMS

1. Using the ratio of successive terms of a sequence, determine the largest real root of the equation: $x^3 - 12x^2 + 9x - 7 = 0$.

2. By analyzing the roots of the auxiliary equation, determine the limiting ratio of successive terms in the sequences obeying the recursion relation: $T_{n+1} = 8T_{n-1} + 3T_{n-2}$.

3. By analyzing the roots of the auxiliary equation, determine the limiting ratio of successive terms of sequences having the recursion relation: $T_{n+1} = -3T_n + T_{n-1} + 8T_{n-2} + 4T_{n-3}$.

4. If $R_n = 5(-1)^n$ and $S_n = F_n$, what is the limiting ratio of terms of the sequence $T_n = R_n + S_n$ ?

5. If

$$R_n = 2^n(n^2 + 3n + 5) \text{ and } S_n = 3S_{n-1} + S_{n-2},$$

with $S_1 = 1$, $S_2 = 5$, find the limiting ratio of $T_n = R_n + S_n$.

6. By analyzing the auxiliary equation, show that the recursion relation

$$T_{n+1} = 3T_n - 7T_{n-1} + 10T_{n-2}$$

governs sequences which do not have a limiting ratio.

7. Find the limiting ratio of terms of a sequence governed by the recursion relation

$$T_{n+1} = 4T_n - 6T_{n-1} + 4T_{n-2} - T_{n-3}.$$

8. If $P_n = 5 \times 3^{n-1}$ and $Q_n = F_n$, determine the limiting ratio of the terms of the sequence $P_n + Q_n$.

9. Find the limiting ratio (if it exists) of a sequence governed by the recursion relation

10. The sequence 2, 1, -2, -1, 2, 1, -2, -1, $\cdots$ evidently does not have a limiting ratio. Determine its linear recursion relation and on the basis of the roots of the auxiliary equation verify that this is what should be expected.

(See page 60 for answers to problems.)

## LINEAR RECURSION RELATIONS

## ANSWERS TO PROBLEMS

### LESSON ONE

1. $a_n = n(n+1)$; $T_{n+3} = 3T_{n+2} - 3T_{n+1} + T_n$
2. $a_n = 3n - 2$; $T_{n+2} = 2T_{n+1} - T_n$
3. $a_n = n^3$; $T_{n+4} = 4T_{n+3} - 6T_{n+2} + 4T_{n+1} - T_n$
4. $T_{6n+k} = 1, 3, 3, 1, 1/3, 1/3$, for $k = 1, 2, 3, 4, 5, 6$, respectively
5. $T_{n+1} = \sqrt{1 + T_n^2}$
6. $T_{n+4} = 4T_{n+3} - 6T_{n+2} + 4T_{n+1} - T_n$
7. $T_{n+1} = aT_n$
8. $T_{n+3} = 3T_{n+2} - 3T_{n+1} + T_n$
9. $T_{2n-1} = a$, $T_{2n} = 1/a$
10. $T_{n+1} = 1/(2 - T_n)$

### LESSON TWO

1. $$T_n = -4 + (7/2)\,2^n$$

First ten terms: 3, 10, 24, 52, 108, 220, 444, 892, 1788, 3580.

2. $$T_n = (13/6)\,3^n + (-3/10)\,5^n$$

3. $$T_n = 17/5 + (4/15)\,6^n$$
$$T_{n+1} = 7T_n - 6T_{n-1}$$

4. $$T_n = -2 + 3\cdot 2^n + (-1/3)\,3^n$$

5. $$T_{n+1} = 3T_n + T_{n-1} - 3T_{n-2}$$
$$T_n = 1/4 + (7/8)(-1)^n + (13/24)\,3^n .$$

6. $$T_n = 5/3 + (1/3)(-1)^n/2^{n-2}$$

7. $$T_n = 5/2 + (9/2)(-1/3)^n$$

8. $$T_n = 2^{n/2}\left[\frac{5 + 3\sqrt{2}}{4} + (-1)^n\,\frac{5 - 3\sqrt{2}}{4}\right]$$

9. $$T_n = 3 + (-1)^n$$

10. $$T_n = \frac{-2+\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\right)^n + \frac{-2-\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}\right)^n.$$

## LESSON THREE

2. $$2(-1)^n$$

3. $$L_{2n} + (-1)^n$$

4. $$L_{4n} + (-1)^n L_{2n} + 1$$

5. $$L_{2n} + (-1)^{n+1}$$

6. $$L_{4n} + (-1)^{n+1} L_{2n} + 1$$

7. $$T_n = \frac{10+\sqrt{5}}{5} r^n + \frac{10-\sqrt{5}}{5} s^n$$

8. $$F_n = 2^{-n+1}\left[n + 5\binom{n}{3} + 5^2\binom{n}{5} + 5^3\binom{n}{7} \cdots\right]$$

9 $$L_n = 2^{-n+1}\left[1 + 5\binom{n}{2} + 5^2\binom{n}{4} + 5^3\binom{n}{6} \cdots\right]$$

10. $$F_{2n+1}.$$

## LESSON FOUR

1. For any modulus m, there are m possible residues $(0, 1, 2, \cdots, m-1)$. Successive pairs may come in $m^2$ ways. Two successive residues determine all residues thereafter. Now in an infinite sequence of residues there is bound to be repetition and hence periodicity.

Since m divides $T_0$, it must by reason of periodicity divide an infinity of members of the sequence.

2. $n = mk$, where m and k are odd. $V_n$ can be written

$$V_n = (r^m)^k + (s^m)^k ,$$

which is divisible by $V_m = r^m + s^m$.

3. $r = 2 + 2i\sqrt{2}$, $s = 2 - 2i\sqrt{2}$.

$$T_n = \left(\frac{2 - 3i\sqrt{2}}{16}\right) r^n + \left(\frac{2 + 3i\sqrt{2}}{16}\right) s^n .$$

4. The auxiliary equation is $(x - 1)^2 = 0$, so that $T_n$ has the form

$$T_n = An \times 1^n + B \times 1^n = An + B .$$

5. $$T_n = 2^n\left[\left(\frac{b - 2a}{4}\right)n + \frac{4a - b}{4}\right] .$$

6. $$T_n = -(-i)^n$$

7. $$T_{n+1} = 5T_n - 6T_{n-1}$$
$$T_n = 2^n + 3^{n-1}$$

8. $$r = \frac{5 + \sqrt{29}}{2}, \qquad s = \frac{5 - \sqrt{29}}{2}$$

$$T_n = \frac{r^n - s^n}{\sqrt{29}} \quad \text{with terms } 1, 5, 26, 135, \cdots$$

$$V_n = r^n + s^n \quad \text{with terms } 5, 27, 140, \cdots$$

9. $$r = \frac{3 + i\sqrt{11}}{2}, \qquad s = \frac{3 - i\sqrt{11}}{2}$$

$$T_n = \left(\frac{33 - 16i\sqrt{11}}{55}\right) r^n + \left(\frac{33 + 16i\sqrt{11}}{55}\right) s^n$$

10. $$T_{n+1} = 5T_n + 2T_{n-1}; \quad T_1 = 3, \quad T_2 = 7 .$$

## LESSON FIVE

1. $$T_{n+1} = 8T_n - 18T_{n-1} + 16T_{n-2} - 5T_{n-3}$$

2. $$T_n = -5/2 + 7 \times 2^n - (7/6)3^n$$

3. $$T_{n+1} = 4T_n - 2T_{n-1} - 3T_{n-2}$$

4. $$T_{n+1} = 2T_n + T_{n-1} - 3T_{n-2} + T_{n-4}$$

5. $$T_n = 12 + \frac{1}{\sqrt{13}}\left(\frac{3+\sqrt{13}}{2}\right)^n - \frac{1}{\sqrt{13}}\left(\frac{3-\sqrt{13}}{2}\right)^n$$

6. $$T_n = (-135/20)(-1)^n + (19/10)(-2)^n + (41/60)3^n$$

7. $$T_{n+1} = 3T_{n-1} + 2T_{n-2}$$

8. $$T_n = -1/3 + 4n - (-2)^n/6$$

9. $T_{n+1} = 3T_n - 3T_{n-1} + T_{n-2}$ and $T_n = 2 + n/2 + 3n^2/2$

10. $T_{n+1} = -T_{n-1}$ and $T_n = \frac{-3-i}{2} i^n + \frac{-3+i}{2}(-i)^n$

## LESSON SIX

1. $$T_{n+1} = 5T_n + 2T_{n-1} - 9T_{n-2} - 5T_{n-3}$$

2. $$T_{n+1} = 5T_n - 4T_{n-1} - 9T_{n-2} + 7T_{n-3} + 6T_{n-4}$$

3. $$T_{n+1} = 5T_n - 7T_{n-1} + 3T_{n-2}$$

4. $$T_{n+4} = 4T_{n+3} - 2T_{n+2} - 5T_{n+1} + 2T_n$$

5. $$T_{n+6} = 2T_{n+5} + 4T_{n+4} - 4T_{n+3} - 6T_{n+2} + T_n$$

6. $T_{n+1} = 7T_n - 17T_{n-1} + 17T_{n-2} - 6T_{n-3}$

7. $T_{n+4} = T_n$ and $T_n = (-1)^n/2 + \frac{-3-5i}{4} i^n + \frac{-3+5i}{4} (-i)^n$

8. $T_{n+1} = 4T_n - 5T_{n-1} + T_{n-2} + 2T_{n-3} - T_{n-4}$

9. $T_{n+1} = 6T_n - 11T_{n-1} + 5T_{n-2} + 4T_{n-3} - 3T_{n-4}$

10. $T_{n+1} = 9T_n - 27T_{n-1} + 25T_{n-2} + 13T_{n-3} - 19T_{n-4} - 6T_{n-5}$ .

## LESSON SEVEN

1. $5n^3 - 4n^2 + 3n - 8$

2. $3n^2 - 8n + 4$ and the Fibonacci sequence: 1, 4, 5, 9, 14, ···

3. $7n^3 + 3n^2 - 5n + 2 + 3 \times 2^n$

4. $4n + 3 + 3(-1)^n$

5. $2n^3 - 3n^2 - n + 5$ and the Fibonacci sequence $4L_n$

6. $5 \times 4^{n-1} + 17n + 19$

7. The Fibonacci sequence 1, 4, 5, 9, 14, 23, ··· and the arithmetic progression $6n + 1$

8. $7 \times 3^{n-1} + n^2/2 + n/2 + 2$

9. The Fibonacci sequence 3, 7, 10, 17, ··· and the polynomial $(7n^2 - 27n + 28)/2$

10. The Fibonacci sequence 5, 11, 16, 27, ··· and $6 \times 2^{n-1}$.

## LESSON EIGHT

1. 11.2556550
2. The roots are 3, and

$$\frac{-3 \pm \sqrt{5}}{2} .$$

Limiting ratio is 3.

3. The roots are -2, -2, r and s. Limiting ratio is -2.
4. The roots of the combined recursion relation will be 1, r, s. Limiting ratio is r.
5. The roots of the combined recursion relation are +2, +2, +2,

$$\frac{3 \pm \sqrt{13}}{2} .$$

The limiting ratio is

$$\frac{3 + \sqrt{13}}{2} = 3.3027756 .$$

6. The roots of the auxiliary equation are 2,

$$\frac{1 \pm \sqrt{19}\, i}{2} .$$

The absolute value of the complex roots is greater than 2. Thus the sequences will not have a limiting ratio.

7. The limiting ratio is 1.
8. The limiting ratio is 3.
9. The roots of the auxiliary equation are 1, $1 \pm i$. Since the absolute value of the complex root is greater than 1, there is no limiting ratio.
10. The recursion relation is $T_{n+1} = -T_{n-1}$ with roots $\pm i$ for the auxiliary equation. Hence, there is no limiting ratio.