A PRIMER FOR THE FIBONACCI NUMBERS: PART VIII

Marjorie Bicknell
A. C. Wilcox High School, Santa Clara, California

SEQUENCES OF SUMS FROM PASCAL'S TRIANGLE

There are many ways to generalize Fibonacci numbers, one way being to consider them as a sequence of sums found from diagonals in Pascal's triangle [1], [2]. Since Pascal's triangle and computations with generating functions are so interrelated with the Fibonacci sequence, we introduce a way to find such sums in this section of the Primer.

1. INTRODUCTION

Some elementary but elegant mathematics solves the problem of finding the sums of integers appearing on diagonals of Pascal's triangle. Writing Pascal's triangle in a left-justified manner, the problem is to find the infinite sequence of sums p/q of binomial coefficients appearing on diagonals p/q for integers p and q, $p + q \geq 1$, $q > 0$, where we find entries on a diagonal p/q by counting up p and right q, starting in the left-most column. (Notice that, while the intuitive idea of "slope" is useful in locating the diagonals, the diagonal 1/2, for example, is not the same as 2/4 or 3/6.) As an example, the sums 2/1 on diagonals formed by going up 2 and right 1 are 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, ... , as illustrated below:

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

Some sequences of sums are simple to find. For example, the sums 0/1 formed by going up 0 and right 1 are the sums of integers appearing in each row, the powers of 2. The sums 0/2 are formed by alternate integers in a row, also powers of 2. The sums 1/1 give the famous Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, ..., defined by $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$. The sums -1/2, found by counting down 1 and right 2, give the Fibonacci numbers with odd subscripts, 1, 2, 5, 13, 34, 89, ..., F_{2n+1}, ... While the problem is not defined for negative "slope" less than or equal to -1 nor for summing columns, the diagonals -1/1 are the same as the columns of the array, and the sum of the first j integers in the nth column is the same as the jth entry in the (n + 1)st column.

To solve the problem in general, we develop some generating functions.

2. GENERATING FUNCTIONS FOR THE COLUMNS OF PASCAL'S TRIANGLE

Here, a generating function is an algebraic expression which lists terms in a sequence as coefficients in an infinite series. For example, by the formula for summing an infinite geometric progression,

$$(1) \qquad \frac{a}{1-r} = a + ar + ar^2 + ar^3 + \ldots, \qquad |r| < 1\,,$$

we can write the generating function for the powers of 2 as

$$(2) \qquad \frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \ldots + 2^n x^n + \ldots\,, \qquad |x| < 1/2\,.$$

Long division gives a second verification that $1/(1 - 2x)$ generates powers of 2, and long division can be used to compute successive coefficients of powers of x for any generating function which follows.

We need some other generating functions to proceed. By summing the geometric progression,

$$(3) \qquad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{k=0}^{\infty} \binom{k}{0} x^k\,, \qquad |x| < 1\,.$$

By multiplying series or by taking successive derivatives of (3), one finds

$$(4) \qquad \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots + kx^{k-1} + \ldots = \sum_{k=0}^{\infty} \binom{k}{1} x^k\,, \quad |x| < 1,$$

$$(5) \qquad \frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + \ldots = \sum_{k=0}^{\infty} \binom{k}{2} x^k\,, \quad |x| < 1.$$

Computation of the nth derivative of (3) shows that

$$\frac{1}{(1-x)^{n+1}} = \sum_{k=0}^{\infty} \binom{k}{n} x^k, \quad n = 0, 1, 2, 3, \ldots, \quad |x| < 1,$$

is a generating function for the integers appearing in the nth column of Pascal's triangle, or equivalently, the column generator for the nth column, where we call the left-most column the zero-th column. As a restatement, the columns of Pascal's triangle give the coefficients of the binomial expansion of $(1 - x)^{-n-1}$, $n = 0, 1, 2, \ldots$, $|x| < 1$, or of $(1 + x)^{-n-1}$ if taken with alternating signs.

3. SOME PARTICULAR SUMS DERIVED USING COLUMN GENERATORS

It is easy to prove that the rows in Pascal's triangle have powers of 2 as their sums: merely let $x = 1$ in $(x + 1)^n$, $n = 0, 1, 2, \ldots$. But, to demonstrate the methods, we work out the sums 0/1 of successive rows using column generators.

First write Pascal's triangle to show the terms in the expansions of $(x + 1)^n$. Because we want the exponents of x to be identical in each row so that we will add the coefficients in each row by adding the column generators, multiply the columns successively by $1, x, x^2, x^3, \ldots$, making

	1					
	$1x$	$1x$				
	$1x^2$	$2x^2$	$1x^2$			
	$1x^3$	$3x^3$	$3x^3$	$1x^3$		
	$1x^4$	$4x^4$	$6x^4$	$4x^4$	$1x^4$	
	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
generators:	$\frac{1}{1-x}$	$\frac{x}{(1-x)^2}$	$\frac{x^2}{(1-x)^3}$	$\frac{x^3}{(1-x)^4}$	$\frac{x^4}{(1-x)^5}$	$\ldots$

Then the sum S of column generators will have the sums 0/1 of the rows appearing as coefficients of successive powers of x. But, S is a geometric progression with ratio $x/(1 - x)$, so by (1),

$$S = \frac{\frac{1}{1-x}}{1 - \frac{x}{1-x}} = \frac{1}{1-2x}, \qquad \text{for} \qquad \left|\frac{x}{1-x}\right| < 1 \quad \text{or} \quad |x| < 1/2,$$

the generating function for powers of 2 given earlier in (2).

If we want the sums 0/2, we sum every other generating function, forming

$$S^* = \frac{1}{1-x} + \frac{x^2}{(1-x)^3} + \frac{x^4}{(1-x)^5} + \ldots,$$

and again sum the geometric progression to find

$$S^* = \frac{1-x}{1-2x} = \frac{1}{1-2x} - \frac{x}{1-2x}$$

$$= (1 + 2x + 4x^2 + 8x^3 + \ldots + 2^n x^n + \ldots)$$

$$- (x + 2x^2 + 4x^3 + \ldots + 2^{n-1}x^n + \ldots)$$

$$= 1 + (x + 2x^2 + 4x^3 + \ldots + 2^{n-1}x^n + \ldots),$$

which again generates powers of 2 as verified above.

We have already noted that the sums 1/1 give the Fibonacci numbers. To use column generators, we must multiply the columns successively by $1, x^2, x^4, x^6, \ldots$, so that the exponents of x will be the same along each diagonal 1/1. The sum S^{**} of column generators becomes

$$S^{**} = \frac{1}{1-x} + \frac{x^2}{(1-x)^2} + \frac{x^4}{(1-x)^3} + \frac{x^6}{(1-x)^4} + \ldots,$$

again a geometric progression, so that

$$S^{**} = \frac{\frac{1}{1-x}}{1 - \frac{x^2}{1-x}} = \frac{1}{1-x-x^2}$$

for

$$\left|\frac{x^2}{1-x}\right| < 1 \qquad \text{or} \qquad |x| < \left(\frac{1+\sqrt{5}}{2}\right)^{-1}.$$

This means that, for $|x|$ less than the positive root of $x^2 + x - 1 = 0$,

$$(6) \qquad \frac{1}{1 - x - x^2} = 1 + 1x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \ldots + F_n x^{n-1} + \ldots$$

For generating functions, we are concerned primarily with the coefficients of x rather than values of x, but a particular example is interesting at this point. Both series (4) and (6) converge when $x = 1/2$; let $x = 1/2$ in those series to form

$$4 = 1 + 2\cdot\frac{1}{2} + 3\cdot\frac{1}{4} + 4\cdot\frac{1}{8} + 5\cdot\frac{1}{16} + 6\cdot\frac{1}{32} + \ldots + n\cdot\frac{1}{2^{n-1}} + \ldots,$$

$$4 = 1 + 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{8} + 5\cdot\frac{1}{16} + 8\cdot\frac{1}{32} + \ldots + F_n\cdot\frac{1}{2^{n-1}} + \ldots,$$

the same result whether we use the natural numbers or the Fibonacci numbers as coefficients of the powers of 1/2 !

Also, $x = 0.1$ in (6) gives, upon division by 100,

$$\begin{array}{l} \frac{1}{89} = 0.0112358 \\ \phantom{\frac{1}{89} = 0.011235}13 \\ \phantom{\frac{1}{89} = 0.0112358}21 \\ \phantom{\frac{1}{89} = 0.01123581}34 \\ \phantom{\frac{1}{89} = 0.011235813}55 \\ \phantom{\frac{1}{89} = 0.0112358132}89 \\ \phantom{\frac{1}{89} = 0.01123581321}\ldots \end{array}$$

the reciprocal of a Fibonacci number with successive Fibonacci numbers making up its decimal expansion.

We are now in a position to solve the general problem of finding the sums p/q.

4. SEQUENCES OF SUMS p/q APPEARING ALONG ANY DIAGONAL

To find the sequence of sums appearing along the diagonals p/1, multiply the columns of Pascal's triangle successively by 1, x^{p+1}, $x^{2(p+1)}$, $x^{3(p+1)}$, ..., so that the exponents of x appearing on each diagonal p/1 will be the same, giving

$$\begin{array}{llllll}
1 & & & & & \\
1x & 1x^{p+1} & & & & \\
1x^2 & 2x^{p+2} & 1x^{2p+2} & & & \\
1x^3 & 3x^{p+3} & 3x^{2p+3} & 1x^{3p+3} & & \\
1x^4 & 4x^{p+4} & 6x^{2p+4} & 4x^{3p+4} & 1x^{4p+4} & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{array}$$

generators: $\dfrac{1}{1-x}\quad \dfrac{x^{p+1}}{(1-x)^2}\quad \dfrac{x^{2p+2}}{(1-x)^3}\quad \dfrac{x^{3p+3}}{(1-x)^4}\quad \dfrac{x^{4p+4}}{(1-x)^5}\quad \cdots$.

The sum S of column generators is a geometric progression, so that

$$(7)\qquad S = \frac{\dfrac{1}{1-x}}{1 - \dfrac{x^{p+1}}{1-x}} = \frac{1}{1 - x - x^{p+1}}\ , \quad p \geq 1, \qquad \left|\frac{x^{p+1}}{1-x}\right| < 1\ ,$$

with S convergent for $|x|$ less than the positive root of $x^{p+1} + x - 1 = 0$. Then, the generating function (7) gives the sums p/1 as coefficients of successive powers of x . [Reader: Show $|x| < 1/2$ is sufficient. Editor.]

In conclusion, the sequence of sums p/q are found by multiplying successive qth columns by 1, x^{p+q}, $x^{2(p+q)}$, $x^{3(p+q)}$, ..., making the sum of column generators be

$$S^* = \frac{1}{1-x} + \frac{x^{p+q}}{(1-x)^{q+1}} + \frac{x^{2p+2q}}{(1-x)^{2q+1}} + \frac{x^{3p+3q}}{(1-x)^{3q+1}} + \cdots\ .$$

Summing that geometric progression yields the generating function

$$S^* = \frac{(1-x)^{q-1}}{(1-x)^q - x^{p+q}}\ , \qquad p + q \geq 1, \qquad q > 0,$$

which converges for $|x|$ less than the absolute value of the root of smallest absolute value of $x^{p+q} - (1-x)^q = 0$ and which gives the sums of the binomial coefficients found along the diagonals p/q as coefficients of successive powers of x . [Reader: Show $|x| < 1/2$ is sufficient. Editor.]

Some references for readings related to the problem of this paper follow but the list is by no means exhaustive. We leave the reader with the problem of determining the properties of particular sequences of sums arising in this paper.

REFERENCES

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