A MOTIVATION FOR CONTINUED FRACTIONS

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This Quarterly is devoted to the study of properties of integers, especially to the study of recurrent sequences of integers. We show below how such sequences and continued fractions arise naturally in the problem of approximating an irrational number to any desired closeness by rational numbers.

We begin with the equation

$$x^2 - x - 1 = 0 \quad . \tag{1}$$

One can easily see that there is a negative root between -1 and 0 and a positive root between 1 and 2, for example by graphing $y = x^2 - x - 1$. We call the positive root r. This number has been known since antiquity as the "golden mean." We now look for a sequence of rational approximations to r.

A rational number is of the form p/q with p and q integers (and $q \neq 0$). We therefore wish two sequences

$$p_1, p_2, p_3, \ldots \qquad \text{and} \qquad q_1, q_2, q_3, \ldots \tag{2}$$

of integers such that the quotients p_n/q_n are approximations which get arbitrarily close to r. It would also be helpful if each new approximation were obtainable simply from previous ones.

We go back to (1) and rewrite it as

$$x = 1 + \frac{1}{x} \quad . \tag{3}$$

This states that if we replace x by r in

$$1 + \frac{1}{x} \tag{4}$$

the result is r and suggests that if we replace x in (4) by an approximation to r we will get another approximation. We now change (3) into the form

$$x_2 = 1 + \frac{1}{x_1} \tag{5}$$

and consider x_1 to be an approximation to r. The relative error of $1/x_1$ is the same as that of x_1 and, if x_1 is positive, the relative error of x_2 (i. e., $1 + 1/x_1$) is lower than that of x_1, since adding 1 increases the number but not the error. It can be shown that x_2 in (5) is a better approximation to r than x_1, if $x_1 > 0$.

We now let our first approximation x_1 be a rational number p_1/q_1 and substitute this in (5) obtaining

$$x_2 = 1 + \frac{1}{(p_1/q_1)} = 1 + \frac{q_1}{p_1} = \frac{p_1 + q_1}{p_1} .$$

We therefore choose p_2 to be $p_1 + q_1$ and q_2 to be p_1 . Similarly, our third approximation is p_3/q_3 with $p_3 = p_2 + q_2$ and $q_3 = p_2$. In general, the (n + 1)st approximation p_{n+1}/q_{n+1} has

(6) $$p_{n+1} = p_n + q_n$$

(7) $$q_{n+1} = p_n .$$

It follows from (7) that $q_n = p_{n-1}$; substituting this in (6) gives

(8) $$p_{n+1} = p_n + p_{n-1} .$$

Since r is between 1 and 2 we use 1 as the first approximation, i.e., we let $p_1 = q_1 = 1$. This means that $p_2 = 2$ and it now follows from (8) that p_n is the Fibonacci number F_{n+1} . Then (7) implies that $q_n = F_n$ and we see that the sequence of quotients F_{n+1}/F_n of consecutive Fibonacci numbers furnishes the desired approximations to the root r of (1). It can be shown that this sequence converges to r in the calculus sense.

We next consider the problem of approximating $s = \sqrt{10}$ in this way. The number s is the positive root of

(9) $$x^2 - 10 = 0 .$$

We write (9) in the forms

$$x^2 - 9 = 1$$

(10)
$$(x - 3)(x + 3) = 1$$
$$(x - 3) = 1/(x + 3)$$
$$x = 3 + 1/(x + 3)$$

and change (10) into

(11)
$$x_{n+1} = 3 + \frac{1}{3 + x_n} \quad .$$

Again, if x_n is a positive approximation to s, it can be seen that x_{n+1} is an approximation with smaller relative error. There is a sequence of rational approximations p_n/q_n with

$$p_{n+1} = 3p_n + 10q_n \,, \qquad q_{n+1} = p_n + 3q_n \quad .$$

Letting the first approximation be 3, i.e., letting $p_1 = 3$ and $q_1 = 1$, we obtain the sequence

$$3/1, \quad 19/6, \quad 117/37, \quad \ldots$$

which can be shown to converge to s.

Equation (11) contains the equations

$$x_2 = 3 + \frac{1}{3 + x_1} \,, \qquad x_3 = 3 + \frac{1}{3 + x_2} \quad .$$

Substituting the first of these into the second gives us

$$x_3 = 3 + \cfrac{1}{6 + \cfrac{1}{3 + x_1}} \quad .$$

If this is substituted into $x_4 = 3 + 1/(3 + x_3)$ and if we let x_1 be 3, we obtain

$$x_4 = 3 + \cfrac{1}{6 + \cfrac{1}{6 + \frac{1}{6}}} \quad .$$

In this way we can write continued fraction expressions for any one of the x_n. Then it is natural to let the infinite continued fraction

$$3 + \cfrac{1}{6 + \cfrac{1}{6 + \cdots}}$$

represent the limit s of the sequences x_n defined by (11) and $x_1 = 3$.

The infinite continued fraction for the root r of $x^2 - x - 1 = 0$ is

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}} \ ,$$

whose elegant simplicity is worthy of the title "golden mean."

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A CURIOUS FORMULA FOR THE GOLDEN SECTION RATIO

A curious formula which relates the Golden Section Ratio $\phi = \frac{1 + \sqrt{5}}{2}$, the imaginary unit $i = \sqrt{-1}$, and e, the base of natural logarithms, is

$$\phi = 2 \cos \left(\frac{\log_e (i^2)}{5i} \right) ;$$

can you prove it? (See J. A. H. Hunter and Joseph S. Madachy, Mathematical Diversions, D. Van Nostrand, Princeton, New Jersey, 1963. Pp. 14-19.)

Formulas also relate the Golden Section Ratio ϕ to trigonometric functions. (See Bicknell and Hoggatt, "Golden Triangles, Rectangles, and Cuboids", pages 75 and 76.) It can be proved that $\sin 18° = 1/2\phi$ and that $\sin 54° = \phi/2$.

Another interesting formula follows which is related to the first problem.

B-18 (Proposed by J. L. Brown, Jr.) Show that

$$F_n = 2^{n-1} \sum_{k=0}^{n-1} (-1)^k \cos^{n-k-1} \frac{\pi}{5} \sin^k \frac{\pi}{10} \ , \quad \text{for } n \geq 1 .$$