A PRIMER FOR THE FIBONACCI SEQUENCE: PART III

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MORE FIBONACCI IDENTITIES FROM MATRICES AND VECTORS

The algebra of vectors and matrices will be further pursued to derive some more Fibonacci identities.

1. THE ALGEBRA OF (TWO-DIMENSIONAL) VECTORS

The two-dimensional vector V is an ordered pair of elements, called scalars, of a field: $V = (a, b)$. (The real numbers, for example, form a field.)

The zero vector, $\emptyset$, is a vector whose elements are each zero; $\emptyset = (0, 0)$.

Two vectors, $U = (a, b)$ and $V = (c, d)$, are equal if and only if their corresponding elements are equal; that is, if and only if $a = c$ and $b = d$.

The vector W, which is the product of a scalar k and a vector $U = (a, b)$, is $W = kU = (ka, kb) = Uk$. We see that if $k = 1$, then $kU = U$. We shall define the additive inverse of U by $-U = (-1)U$.

The vector W, which is the vector sum of two vectors $U = (a, b)$ and $V = (c, d)$ is

$$W = U + V = (a, b) + (c, d) = (a + c, b + d).$$

The vector $W = U - V = U + (-V)$, which defines subtraction.

The only binary multiplicative operation between two vectors, $U = (a, b)$ and $V = (c, d)$, considered here is the scalar or inner product,

$$U \cdot V = (a, b) \cdot (c, d) = ac + bd,$$

which is a scalar.

2. A GEOMETRIC INTERPRETATION OF A TWO-DIMENSIONAL VECTOR

One interpretation of the vector $U = (a, b)$ is a directed line segment from the origin (0, 0) to the point (a, b) in a rectangular coordinate system. Every vector, except the zero vector $\emptyset$, will have the direction from the origin to the point (a, b) and a magnitude or length, $|U| = \sqrt{a^2 + b^2}$. The zero vector $\emptyset$ has a zero magnitude and no defined direction.

The inner or scalar product of two vectors, $U = (a, b)$ and $V = (c, d)$ can be shown to equal

$$U \cdot V = |U||V| \cos \theta,$$

where θ is the angle between the two vectors.

3. TWO-BY-TWO MATRICES AND TWO-DIMENSIONAL VECTORS

If U = (a, b) is written as (a b), then U is a 1 x 2 matrix which we shall call a row-vector. Similarly, if U = (a, b) is written vertically, then U becomes a 2 x 1 matrix which we shall call a column-vector.

The matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

for example, can be considered as two row-vectors $R_1 = (a \;\; b)$ and $R_2 = (c \;\; d)$ in special position, or, as two column-vectors $C_1 = \begin{pmatrix} a \\ c \end{pmatrix}$ and $C_2 = \begin{pmatrix} b \\ d \end{pmatrix}$ in special position.

The product W of a matrix A and a column-vector X is a column-vector X',

$$W = AX = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} .$$

Thus matrix A, operating upon the vector X, yields another vector, X'. The zero vector Ø is transformed into the zero vector again. In general, the direction and magnitude of vector X are different from those of vector X'.

4. THE INVERSE OF A TWO-BY-TWO MATRIX

If the determinant D(A) of a two-by-two matrix A is non-zero, then there exists a matrix A^{-1}, the inverse of matrix A, such that $A^{-1}A = AA^{-1} = I$. From the equation AX = X' or pair of equations ax + by = x' and cx + dy = y', one can solve for the variables x and y provided that $D(A) = ad - bc \neq 0$. Suppose this has been done, letting $D = D(A) \neq 0$, so that

$$\frac{d}{D}x' - \frac{b}{D}y' = x$$

$$\frac{-c}{D}x' + \frac{a}{D}y' = y .$$

Thus the matrix B, such that BX' = X, is given by

$$B = \begin{pmatrix} \frac{d}{D} & \frac{-b}{D} \\ \frac{-c}{D} & \frac{a}{D} \end{pmatrix}, \qquad D \neq 0.$$

It is easy to verify that BA = AB = I. Thus B is A^{-1}, the inverse matrix to matrix A. The inverse of the Q matrix is $Q^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$.

5. FIBONACCI IDENTITY USING THE Q MATRIX

Suppose we prove, recalling that $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$,

that $F_1 + F_2 + \dots + F_n = F_{n+2} - 1$.

It is easy to establish by mathematical induction that

$$(I + Q + Q^2 + \dots + Q^n)(Q - I) = Q^{n+1} - I .$$

If $(Q - I)$ has an inverse $(Q - I)^{-1}$, then multiplying on each side yields

$$I + Q + Q^2 + \dots + Q^n = (Q^{n+1} - I)(Q - I)^{-1} .$$

It is easy to verify that Q satisfies the matrix equation $Q^2 = Q + I$. Thus $(Q - I)Q = Q^2 - Q = I$ and $(Q - I)^{-1} = Q$. Therefore

$$Q + Q^2 + \dots + Q^n = Q^{n+2} - (Q + I) = Q^{n+2} - Q^2.$$

Equating elements in the upper right in the above matrix equation yields

$$F_1 + F_2 + \dots + F_n = F_{n+2} - F_2 = F_{n+2} - 1.$$

6. THE CHARACTERISTIC POLYNOMIAL OF A MATRIX A

In section 3, we discussed the transformation $AX = X'$. Generally the direction and magnitude of vector X are different from those of vector X'. If we ask which vectors X have their directions unchanged, we are led to the equation $AX = \lambda X$, where λ is a scalar. This can be rewritten as

$$(A - \lambda I)X = \emptyset .$$

Since we want $|X| \neq 0$, the only possible solution occurs when $D(A - \lambda I) = 0$. This last equation is called the <u>characteristic equation</u> of matrix A. The values of λ are called characteristic values or eigenvalues and the associated vectors are the characteristic vectors of matrix A. The characteristic polynomial of A is $D(A - \lambda I)$.

The characteristic equation for the Q matrix is $\lambda^2 - \lambda - 1 = 0$. The Hamilton-Cayley theorem states that a matrix satisfies its own characteristic equation, so that for the Q matrix

$$Q^2 - Q - I = 0.$$

Of course, this can be rewritten as $Q^2 = Q + I$, in which form we will use the matrix equation in the next section.

7. SOME MORE IDENTITIES

Let $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, which satisfies $Q^2 = Q + I$. Thus, since $Q^0 = I$,

(1) $$Q^{2n} = (Q^2)^n = (Q + I)^n = \sum_{i=0}^{n} \binom{n}{i} Q^i$$

Equating elements in the upper right yields

$$F_{2n} = \sum_{i=0}^{n} \binom{n}{i} F_i .$$

From (1)

$$Q^p Q^{2n} = \sum_{i=0}^{n} \binom{n}{i} Q^{i+p} ,$$

which gives

$$F_{2n+p} = \sum_{i=0}^{n} \binom{n}{i} F_{i+p}$$

for $n \geq 0$ and integral p.

From part II, $Q^n = F_n Q + F_{n-1} I$, so that

$$Q^{mn+p} = \sum_{i=0}^{m} \binom{m}{i} Q^{i+p} F_n^i F_{n-1}^{m-i} .$$

Equating elements in the upper right of the above matrix equation gives

$$F_{mn+p} = \sum_{i=0}^{m} \binom{m}{i} F_{i+p} F_n^i F_{n-1}^{m-i} ,$$

with $m \geq 0$, and for any integral p and n.