A PRIMER FOR THE FIBONACCI NUMBERS: PART XI

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MULTISECTION GENERATING FUNCTIONS FOR THE COLUMNS OF PASCAL'S TRIANGLE

1. INTRODUCTION

Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

be the generating function for the sequence $\{a_n\}$. Often one desires generating functions which multisect the sequence $\{a_n\}$,

$$G_{i}(x) = \sum_{j=0}^{\infty} a_{i+mj} x^{j}$$
, (i = 0, 1, 2, ..., m-1).

For the bisection generating functions the task is easy. Let

$$H_1(x^2) = \frac{f(x) + f(-x)}{2}$$
,

$$H_2(x^2) = \frac{f(x) - f(-x)}{2x}$$
;

then clearly $H_1(x^2)$ and $H_2(x^2)$ contain only even powers of x so that

$$H_1(x) = \sum_{n=0}^{\infty} a_{2n} x^n$$
 and $H_2(x) = \sum_{n=0}^{\infty} a_{2n+1} x^n$

are what we are looking for.

Let us illustrate this for the Fibonacci sequence. Here

$$f(x) = \frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n$$
;

then

$$H_1(x) = \frac{x}{1 - 3x + x^2} = \sum_{n=0}^{\infty} F_{2n}x^n$$

and

$$H_2(x) = \frac{1-x}{1-3x+x^2} = \sum_{n=0}^{\infty} F_{2n+1}x^n$$
.

Exercise: Find the bisection generating functions for the Lucas sequence.

Let us find the general multisecting generating functions for the Fibonacci sequence, using the method of H. W. Gould (See [1].). The Fibonacci sequence enjoys the Binet Form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$
, $\alpha = \frac{1 + \sqrt{5}}{2}$, $\beta = \frac{1 - \sqrt{5}}{2}$.

Let f(x) = 1/(1 - x); then

$$\sum_{n=0}^{\infty} F_{mn+j} x^{n} = \frac{\alpha^{j} f(\alpha^{m} x) - \beta^{j} f(\beta^{m} x)}{\alpha - \beta}$$

$$= \frac{1}{\alpha - \beta} \left(\frac{\alpha^{j}}{1 - \alpha^{m} x} - \frac{\beta^{j}}{1 - \beta^{m} x} \right)$$

$$= \frac{\frac{\alpha^{j} - \beta^{j}}{\alpha - \beta} + (\alpha \beta)^{j} \frac{\alpha^{m-j} - \beta^{m-j}}{\alpha - \beta} x}{1 - (\alpha^{m} + \beta^{m})x + (\alpha \beta)^{m} x^{2}}$$

$$= \frac{F_{j} + (-1)^{j} F_{m-j} x}{1 - L x + (-1)^{m} x^{2}}, \quad (j = 0, 1, 2, ..., m-1),$$

since
$$\alpha \beta = -1$$
, $\alpha^m + \beta^m = L_m$, and $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$.

Exercise: Find the general multisecting generating function for the Lucas sequence.

The same technique can be used on any sequence having a Binet Form. The general problem of multisecting a general sequence rapidly becomes very complicated according to Riordan [2], even in the classical case.

2. COLUMN GENERATORS OF PASCAL'S TRIANGLE

The column generators of Pascal's left-justified triangle [3], [4], [5], are

$$G_k(x) = \frac{x^k}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} {n \choose k} x^n, \quad k = 0, 1, 2, \dots$$

We now seek generating functions which will m-sect these,

$$G_{i}(m, k; x) = \sum_{n=0}^{\infty} {i+k+mn \choose k} x^{n+k+1}, (i = 0,1,..., m-1).$$

We first cite an obvious little lemma.

$$\underbrace{\text{Lemma 1:}}_{k} \qquad \binom{n}{k} = \sum_{j=1}^{m} \binom{n-j}{k-1} + \binom{n-m}{k}.$$

Definition: Let $G_{i,k}(x)$, i = 0, 1, 2, ..., m - 1, be the m generating functions

$$G_{i,k}(x) = \sum_{n=0}^{\infty} {i+k+mn \choose k} x^{i+mn+k}$$

Lemma 2:

$$G_{i,k+1}(x) = \frac{xG_{i,k}(x) + x^2G_{i-1,k}(x) + \dots + x^mG_{i-m+1,k}(x)}{1 - x^m}$$

The proof follows easily from Lemma 1.

Let

$$(1 + x + x^2 + ... + x^{m-1})^n = \sum_{j=0}^{n(m-1)} {n \choose j}_m x^j$$

define the row elements of the m-nomial triangle. Further, let

$$f_{i}(m,k; x) = \sum_{j=0}^{k} {k \choose i + jm}_{m} x^{j}, i = 0, 1, ..., m-1,$$

where j is such that $i + jm \le k(m-1)$. These are multisecting polynomials for the rows of the m-nomial triangle. Now, we can state an interesting theorem:

Theorem: For i = 0, 1, 2, ..., m - 1,

$$G_{i}(m,k; x) = \frac{x^{k+1} f_{i}(m,k; x)}{(1-x)^{k+1}}$$
.

Proof: Recall first that the m-nomial coefficients obey

$$\binom{n}{r}_{m} = \binom{n-1}{r}_{m} + \binom{n-1}{r-1}_{m} + \cdots + \binom{n-1}{r-m+1}_{m}$$

where the lower arguments are non-negative and less than or equal to n(m-1). Clearly, for k=0, from the definition just before Lemma 2,

$$G_{i,0}(x) = \frac{x^i}{1-x^m}$$
, $i = 0, 1, 2, ..., m-1$.

Assume now that

$$G_{i,k}(x) = \frac{x^{k+i} f_{i(m,k; x^{m})}}{(1 - x^{m})^{k+1}}$$

for i = 0, 1, 2, 3, ..., (m - 1). From Lemma 2,

$$G_{i,k+1}(x) = \frac{xG_{i-1,k}(x) + ... + x^{m}G_{i-m+1,k}(x)}{1 - x^{m}}$$

Thus.

$$G_{i,k+1}(x) = \frac{\sum_{s=0}^{m-1} \left(\sum_{j=0}^{k} \binom{k}{i-s+jm}\right)_{m} x^{k+(i-s)+s+jm+1}}{(1-x^{m})^{k+2}}$$

$$= \frac{\sum_{j=0}^{m-1} \binom{k}{i-s+jm}_{m}}{(1-x^{m})^{k+2}}$$

$$= \frac{x^{k+1+i} \sum_{j=0}^{m-1} \binom{k+1}{i+jm}_{m} x^{jm}}{(1-x^{m})^{k+2}}$$

$$= \frac{x^{k+1+i} \int_{j=0}^{m-1} \binom{k+1}{i+jm}_{m} x^{jm}}{(1-x^{m})^{k+2}}$$

$$= \frac{x^{k+1+i} \int_{j=0}^{m-1} \binom{m+k}{m}_{m} x^{m}}{(1-x^{m})^{k+2}}$$

This completes the induction.

The x^{k+1+i} merely position the column generators. Here the non-zero entries are separated by m-1 zeros. To get rid of the zeros, let

$$G_{i}(m,k; x) = \frac{x^{k+i} f_{i}(m,k; x)}{(1-x)^{k+1}}$$

for i = 0, 1, 2, ..., m-1. This concludes the proof of the theorem.

If we write this in the form

$$G_{\underline{i}}(m,k; x) = \sum_{j=0}^{\infty} {i+jm+k \choose k}^{x^{j+k+1}} = \frac{\sum_{j=0}^{k} {k \choose i+jm}_{m} x^{k+i+j}}{(1-x)^{k+1}}$$

it emphasizes the relation of the multisection of the kth column of Pascal's triangle and the multisection of the kth row of the m-nomial triangle.

Lemma 3:
$$\binom{n}{k} = \sum_{j=0}^{r} \binom{r}{j} \binom{n-r}{k-j}$$

This is easy to prove by starting with

(A)
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$= \binom{n-2}{k} + \binom{n-2}{k-1} + \binom{n-2}{k-1} + \binom{n-2}{k-2}$$

$$= 1 \cdot \binom{n-2}{k} + 2 \cdot \binom{n-2}{k-1} + 1 \cdot \binom{n-2}{k-2}.$$

Apply (A) to each term on the right repeatedly.

Now let $H_i(m,k; x)$ m-sect the kth column of Pascal's triangle (i = 0, 1, 2, ..., m - 1); then, using Lemma 3, it follows that

Lemma 4:
$$H_{\mathbf{j}}(m,k; x) = \frac{x}{1-x} \sum_{j=1}^{m} {m \choose j} H_{\mathbf{j}}(m,k-j; x) .$$

The results using the method of Polya for small m and i seem to indicate the following (See [3].).

Theorem: The generating functions for the rising diagonal sums of the rows of Pascal's triangle i + jm (all other rows are deleted) are given by

$$H_{i}(x) = \frac{(1+x)^{i}}{1-x(1+x)^{m}}, i = 0, 1, ..., m-1.$$

Exercise: Show that

$$\sum_{i=0}^{m-1} x^{i} H_{i}(x^{m}) = \frac{1}{1 - x(1 + x^{m})}.$$

This is a necessary condition which now makes the theorem plausible. These are the generalized Fibonacci numbers obtained as rising diagonal sums from Pascal's triangle, beginning in the left-most column and going over l and up m. (See [3]) The theorem is proved by careful examination of its meaning with regards to Pascal's triangle as follows:

$$\frac{(1+x)^{\frac{1}{n}}}{1-x(1+x)^{\frac{1}{m}}} = \sum_{n=0}^{\infty} x^{n}(1+x)^{mn+\frac{1}{n}} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} {n(n-j)+j \choose j} x^{n},$$

$$i = 0, 1, 2, \ldots, m - 1$$
. Recall that $\binom{n}{k} = 0$ if $0 \le n < k$.

ILLUSTRATION

$$n = 0 x^{0}(1 + x)^{0+1} = 1 + x$$

$$n = 1 x^{1}(1 + x)^{2+1} = x + 3x^{2} + 3x^{3} + x^{4}$$

$$n = 2 x^{2}(1 + x)^{4+1} = x^{2} + 5x^{3} + 10x^{4} + 10x^{5} + 5x^{6} + x^{7}$$

$$n = 3 x^{3}(1 + x)^{6+1} = x^{3} + 7x^{4} + 21x^{5} + \dots$$
Sum:
$$1 + 2x + 4x^{2} + 9x^{3} + 19x^{4} + \dots$$

Here, m = 2 and i = 1. Now, write a left-justified Pascal's triangle. Form the sequence of sums of elements found by beginning in the left-most column and proceeding right one and up 2 throughout the array: 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, Notice that the coefficients of successive powers of x give every other term in that sequence.

The general problem of finding generating functions which multisect the column generators of Pascal's triangle has been solved by Nilson [6], although interpretation of the numerator polynomial coefficients has not been achieved as in our last few theorems.

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