A-CASSINI POLYNOMIAL SEQUENCES AND APPLICATIONS

ROGER C. ALPERIN

Abstract. We examine some polynomial solutions to the A-Cassini relation. We show how these solutions are related to trace polynomials, lengths of the diagonals of a regular polygon, and recurrences given by the characteristic polynomial of certain tridiagonal matrices.

1. A-Cassini Relation and Its Sequences

In this paper, we extend aspects of [2], [1] with an emphasis on polynomial solutions.

We consider the non-linear recurrence

$$g_{n+1}g_{n-1} = g_n^2 - A$$

with non-zero initial values $g_0 = a$, $g_1 = b$, and given A. Thus, $g_2 = \frac{b^2-A}{a}$ and $\frac{g_2+g_0}{g_1} = \frac{a^2+b^2-A}{ab}$.

Theorem 1.1. *Let $\mu = \frac{a^2+b^2-A}{ab}$. Then for $n \geq 2$, $g_n = \mu g_{n-1} - g_{n-2}$.*

Proof. Let $\mu_k = \frac{g_k+g_{k-2}}{g_{k-1}}$. Suppose by induction that $\mu_k = \mu$ and $g_k = \mu g_{k-1} - g_{k-2}$ for $2 \leq k \leq n$. Then,

$$\begin{aligned}
\mu_{n+1} - \mu_n &= \frac{g_{n+1}+g_{n-1}}{g_n} - \frac{g_n+g_{n-2}}{g_{n-1}} \\
&= \frac{(g_{n+1}+g_{n-1})g_{n-1} - (g_n+g_{n-2})g_n}{g_{n-1}g_n} \\
&= \frac{g_{n+1}g_{n-1} + g_{n-1}^2 - g_n^2 - g_{n-2}g_n}{g_{n-1}g_n} \\
&= \frac{g_n^2 - A + g_{n-1}^2 - g_n^2 - g_{n-1}^2 + A}{g_{n-1}g_n} \\
&= 0.
\end{aligned}$$

Hence, $\mu_{n+1} = \mu$ and thus, $g_{n+1} = \mu g_n - g_{n-1}$. Therefore, we have proved the result. □

A polynomial in $x, y, \ldots$ and their inverses $\frac{1}{x}, \frac{1}{y}, \ldots$ is called a Laurent polynomial.

Corollary 1.2. *For indeterminates a and b, $g_n = \mu g_{n-1} - g_{n-2}$ is a Laurent polynomial in a and b.*

Proof. As shown above, g_2 has denominator a. Since μ has denominator ab, the result that g_n has denominator $a^{n-1}b^{n-2}$ for $n \geq 2$ follows by induction. □

Proposition 1.3. *Suppose that $g_{n+1} = Mg_n - g_{n-1}$ with non-zero initial values $g_0 = a$, $g_1 = b$, and given M. Let $A = a^2 + b^2 - Mab$. Then, this sequence satisfies the A-Cassini relation $g_{n+1}g_{n-1} = g_n^2 - A$.*

Proof. By Corollary 1.2, the solution to the A-Cassini relation with $A = a^2 + b^2 - Mab$ and with initial values $h_0 = a$ and $h_1 = b$ is $h_{n+1} = Mh_n - h_{n-1}$, since $\mu = M$. Thus, $h_n = g_n$ for all n. □

The universal A-Cassini polynomials are $u_0 = 1$, $u_1 = t$, and $u_n = tu_{n-1} - u_{n-2}$, for $n \geq 2$. The polynomial u_n is of degree n.

We have the following representation theorem in terms of the universal polynomials.

Theorem 1.4. *Let $A = a^2 + b^2 - Mab$ for given a, b, and M. Then, the sequence satisfying the A-Cassini relation $g_{n+1}g_{n-1} = g_n^2 - A$ with $g_0 = a$ and $g_1 = b$ can be expressed as*

$$g_{n+1} = bu_n(M) - au_{n-1}(M), \text{ for } n \geq 1.$$

Proof. The base case is $bu_1(M) - au_0 = bM - a = g_2$. By induction,

$$\begin{aligned} g_{n+1} &= Mg_n - g_{n-1} \\ &= M(bu_{n-1}(M) - au_{n-2}(M)) - bu_{n-2}(M) + au_{n-3}(M) \\ &= b(Mu_{n-1}(M) - u_{n-2}(M)) - a(Mu_{n-2}(M) - u_{n-3}(M)) \\ &= bu_n(M) - au_{n-1}(M). \end{aligned}$$

□

1.1. Other Identities for u_k. [6]

Theorem 1.5. *Let h and k be nonnegative integers such that $h \leq k$. Then, $u_h u_{k-1} - u_k u_{h-1} = u_{k-h-1}$.*

Proof.

$$\begin{aligned} u_h u_{k-1} - u_k u_{h-1} &= (tu_{h-1} - u_{h-2})u_{k-1} - u_k u_{h-1} \\ &= tu_{h-1}u_{k-1} - u_{h-2}u_{k-1} - u_k u_{h-1} \\ &= u_{h-1}(tu_{k-1} - u_k) - u_{h-2}u_{k-1} \\ &= u_{h-1}u_{k-2} - u_{k-1}u_{h-2}. \end{aligned}$$

Continuing in this way, we have

$$\begin{aligned} u_h u_{k-1} - u_k u_{h-1} &= u_{h-1}u_{k-2} - u_{k-1}u_{h-2} \\ &= \dots \\ &= u_1 u_{k-h} - u_{k-h+1}u_0 \\ &= tu_{k-h} - u_{k-h+1} \\ &= u_{k-h-1}. \end{aligned}$$

□

Corollary 1.6. *Let h and k be nonnegative integers such that $h \leq k$. Then,*

$$u_h u_k = \sum_{i=0}^{h} u_{k-h+2i}.$$

Proof. From Theorem 1.5, $u_h u_k = u_{k+1}u_{h-1} + u_{k-h}$ for $h \leq k$. Using this identity repeatedly, we lower the subscript on u_{h-1} and increase the subscript on u_{k+1} to get $u_h u_k = u_{k+2}u_{h-2} + u_{k-h+2} + u_{k-h}$. Repeat this process until we get the desired formula. □

Proposition 1.7. *Let $n \geq 4$ be an integer. Then, $u_n = (t^2 - 2)u_{n-2} - u_{n-4}$.*

Proof.

$$\begin{aligned} u_n = tu_{n-1} - u_{n-2} &= t(tu_{n-2} - u_{n-3}) - u_{n-2} \\ &= t^2 u_{n-2} - tu_{n-3} - u_{n-2} \\ &= t^2 u_{n-2} - (u_{n-2} + u_{n-4}) - u_{n-2} \\ &= (t^2 - 2)u_{n-2} - u_{n-4}. \end{aligned}$$

□

Corollary 1.8. *$u_k(t)$ is an even (odd) function of t if and only if k is even (odd).*

Proof. It follows immediately from the previous Proposition that u_k is even for k even and from the linear recursion $u_{k+1} = tu_k - u_{k-1}$ it easily follows that u_{k+1} is odd for $k+1$ odd. □

The next result follows immediately by induction.

Corollary 1.9. *For $i \geq 0$, $u_i(2) = i + 1$.*

2. Chebyshev Polynomials

We express the Chebyshev polynomials using the results above.

2.0.1. *First Kind.* The Chebyshev polynomials T_n have the recurrence $T_0 = 1$, $T_1 = x$, and $T_{n+1} = 2xT_n - T_{n-1}$ for $n \geq 1$. Letting $a = 1$, $b = x$, $M = 2x$, and $M = 2x = (1 + x^2 + A)/x$, it follows that $A = x^2 - 1$. Using the representation theorem 1.4 above, we have $T_{n+1}(x) = xu_n(2x) - u_{n-1}(2x) = \frac{1}{2}(2xu_n(2x) - 2u_{n-1}(2x)) = \frac{1}{2}(u_{n+1}(2x) - u_{n-1}(2x))$.

The polynomial formula for these Chebyshev polynomials are well-known [5]:

$$T_n(x) = \frac{n}{2} \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \frac{1}{n-r} \binom{n-r}{r} (2x)^{n-2r}.$$

2.0.2. *Second Kind.* The recurrence relation for the Chebyshev polynomials of the second kind is determined from $a = 1$, $b = 2x$, and $M = 2x$. Hence, $M \cdot 2x = 1 + 4x^2 - A$ and thus, $A = 4x^2 - 4x + 1 = (1 - 2x)^2$. $U_{n+1}(x) = 2xu_n(2x) - u_{n-1}(2x) = u_{n+1}(2x)$. Thus, using the representation identity above, we have $T_{n+1}(x) = \frac{1}{2}(U_{n+1}(x) - U_{n-1}(x))$.

The polynomial formulas for these Chebyshev polynomials are well-known [5]:

$$U_n(x) = \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \binom{n-r}{r} (2x)^{n-r}.$$

Corollary 2.1. *For $n \geq 1$, $u_n(2t) = U_n(t)$. The roots of u_n are $2\cos(\frac{k\pi}{n+1})$ for $k = 1, \ldots, n$.*

3. Trace Polynomials

Consider a diagonalizable matrix $Z \in SL_2(\mathbb{C})$ with trace $\tau = t + t^{-1}$. Then, the trace of Z^n is $(t^n + t^{-n})$. The Cassini relation is

$$(t^{n+1} + t^{-n-1})(t^{n-1} + t^{-n+1}) - (t^n + t^{-n})^2 = t^2 + t^{-2} - 2.$$

Thus, $A = 4 - (t + t^{-1})^2 = 4 - \tau^2$.

For $a = 2$ and $b = \tau$, we have $M = \frac{a^2+b^2-A}{ab} = \frac{4+\tau^2+\tau^2-4}{2\tau} = \tau$. Then, the sequence of traces is $tr(Z^{n+1}) = g_{n+1} = \tau g_n - g_{n-1}$.

We can express these as $g_{n+1}(\tau) = \tau u_n(\tau) - 2u_{n-1}(\tau) = u_{n+1}(\tau) - u_{n-1}(\tau) = 2T_{n+1}(\tau/2)$.

The first few trace polynomials are

$$2,\ \tau,\ \tau^2-2,\ \tau^3-3\tau,\ \tau^4-4\tau^2+2,\ \tau^5-5\tau^3+5\tau.$$

Using the formula for T_n, the trace polynomials have the formula

$$2T_n(\tau/2)=n\sum_{r=0}^{\lfloor n/2\rfloor}(-1)^r\frac{1}{n-r}\binom{n-r}{r}\tau^{n-2r}$$

for $n>0$.

4. Diagonals

Consider the diagonals of a regular polygon of radius 1 having n sides with vertices at the nth roots of unity. The kth vertex is $\zeta_k=\cos(\frac{2k\pi}{n})+I\sin(\frac{2k\pi}{n})$ for $k=0,\ldots,n-1$. Using the Pythagorean formula and measuring from the initial vertex to the kth vertex, the diagonals have length $d_k^2=(\cos(\frac{2k\pi}{n})-1)^2+\sin(\frac{2k\pi}{n})^2=2-2\cos(\frac{2k\pi}{n})$; for $k=1$, d_1 is the side length. Since $\cos(2t)=\cos(t)^2-\sin(t)^2=1-2\sin^2(t)$, the formula above can be rewritten as [6]

$$d_k=2\sin\left(\frac{k\pi}{n}\right).$$

4.1. **Reduced Diagonals.** Consider the reduced diagonals $r_k=\frac{d_k}{d_1}$; if needed, we denote this as $r_k(n)$ to clarify that we are dealing with the regular n-sided polygon.

Using Ptolemy's relation for the lengths of the sides of an inscribed quadrilateral on the circle and its relation to its diagonals, we find the relations $d_hd_{k-1}=d_1d_{k-h}+d_kd_{h-1}$ for $h\le k$. This can be rewritten symmetrically as

$$r_{k-h}=r_hr_{k-1}-r_kr_{h-1}.$$

Let $h=k-1$. Then,

$$r_kr_{k-2}=r_{k-1}^2-1.$$

Since $r_1=1$, $r_3=r_2^2-1$, $r_4=\frac{r_3^2-1}{r_2}=\frac{(r_2^2-1)^2-1}{r_2}=r_2^3-2r_2$, and $r_5=\frac{r_4^2-1}{r_3}=\frac{(r_2^3-2r_2)^2-1}{r_2^2-1}=\frac{r_2^2(r_2^4-4r_2^2+4)-1}{r_2^2-1}=r_2^4-3r_2^2+1$.

Corollary 4.1. $r_{n+1}=u_n(r_2)$ *for* $n\ge 1$.

The next result follows immediately from previous corollaries.

Corollary 4.2. *For* $m>n+1\ge 2$,

$$\frac{\sin(\frac{(n+1)\pi}{m})}{\sin(\frac{\pi}{m})}=\prod_{k=1}^{n}\left(\frac{\sin(\frac{2\pi}{m})}{\sin(\frac{\pi}{m})}-2\cos\left(\frac{k\pi}{n+1}\right)\right).$$

5. Göbels and Junge-Hoggatt Relations

For an n by n matrix B, its characteristic polynomial is $f_B(t)=\det(B-tI)$. Let $g_B(t)=\det(I-tB)$. Since $t^{-1}(B-tI)=t^{-1}B-I=-(I-t^{-1}B)$, $t^{-n}f_B(t)=(-1)^ng_B(t^{-1})$.

In connection with problems related to symmetric polynomials [3], the author considers $w_n(t)=\det(I-tW_n)$, where the n by n matrix $W_n(i,j)=1$, if $i+j\ge n+1$, and otherwise 0. Then, $w_0=1$, $w_1=1-t$, $w_2=1-t-t^2$, $w_3=1+t^3-t^2-2t$, ….

In counting certain sequences generated by reflections [4], the authors consider $v_n(t)=\det(V_n-tI)$, where $V_n(i,j)=1$ if $i+j\le n+1$ (denoted D_n in [4]).

Now, W_n and V_n are similar using the matrix $T = T^{-1}$, where $T(i,j) = 1$ if $i + j = n + 1$, else 0. By the remarks above, we have the following result.

Proposition 5.1. *Let n be a nonnegative integer. Then, $w_n(t) = (-1)^n t^n v_n(t^{-1})$.*

Together with Section 4 and the results shown in [3], we have the following result.

Proposition 5.2. *Let $n \geq 1$. Then,*

$$\begin{aligned} w_{n+1} &= (2 + (-1)^{n+1}t)w_n - w_{n-1} \\ \textit{and } w_{n+1} &= (-1)^{\lceil (n+1)/2 \rceil} u_{n+1} + (-1)^{\lceil n/2 \rceil} u_n. \end{aligned}$$

The signs on the expressions for w_{n+1} have period 4, $(-1,1)$, $(-1,-1)$, $(1,-1)$, $(1,1)$ for $n \geq 0$.

Corollary 5.3. *Let $P_0 = -1$, $P_1 = -1$, and $P_{k+1} = \frac{w_{k+1} - tw_k - w_{k-1}}{2t}$ for $k \geq 1$. Then, $P_{2n} = P_{2n-1}$ for $n \geq 1$.*

Proof.

$$\begin{aligned} P_{2n} - P_{2n-1} &= \frac{w_{2n} - tw_{2n-1} - w_{2n-2}}{2t} - \frac{w_{2n-1} - tw_{2n-2} - w_{2n-3}}{2t} \\ &= \frac{w_{2n} - (t+1)w_{2n-1} + (t-1)w_{2n-2} + w_{2n-3}}{2t} \\ &= \frac{(2+t)w_{2n-1} - w_{2n-2} - (t+1)w_{2n-1} + (t-1)w_{2n-2} + w_{2n-3}}{2t} \\ &= \frac{w_{2n-1} + (t-2)w_{2n-2} + w_{2n-3}}{2t} \\ &= 0. \end{aligned}$$

□

Next, we consider the sequence $Q_k = P_{2k} = P_{2k-1}$. Here is an expression we use in the following theorem.

$$\begin{aligned} Q_k = P_{2k} &= \frac{w_{2k} - tw_{2k-1} - w_{2k-2}}{2t} \\ &= \frac{w_{2k-1} - w_{2k-2}}{t} \\ &= -\frac{u_{2k-1} + u_{2k-3}}{t} \\ &= -u_{2k-2}. \end{aligned}$$

Theorem 5.4. *The sequence $Q_k(t)$ is A-Cassini with $A = t^2$. Hence with $a = b = -1$ and $\mu = 2 - t^2$,*

$$Q_{k+1}(t) = -u_k(2 - t^2) + u_{k-1}(2 - t^2).$$

Proof. The second part follows immediately from the first part and Theorem 1.4. The first part uses some of the identities for u_k in Section 1.1.

Let $C_k = Q_{k+1}Q_{k-1} - Q_k^2$. Then,

$$\begin{aligned}
C_k &= Q_{k+1}Q_{k-1} - Q_k^2 = u_{2k}u_{2k-4} - u_{2k-2}^2 \\
&= \sum_{i=0}^{2k-4} u_{4+2i} - \sum_{i=0}^{2k-2} u_{2i} = \sum_{i=2}^{2k-2} u_{2i} - \sum_{i=0}^{2k-2} u_{2i} \\
&= -u_0 - u_2 = -1 - (t^2-1) = -t^2.
\end{aligned}$$

Hence, $A = t^2$. □

Consider the k by k matrix M_k with i-j entry $\min(i,j)$. It is straightforward to verify that $W_k^2 = M_k$.

Corollary 5.5. *Let $k \geq 0$. Then, $Q_{k+1} = (-1)^{k+1}\det(I - t^2 M_k)$.*

Proof.

$$\begin{aligned}
\det(I - t^2 M_k) &= \det(I - t^2 W_k^2) \\
&= \det(I - tW_k)\det(I + tW_k) \\
&= w_k(t)w_k(-t) \\
&= (u_k(t) - u_{k-1}(t))(u_k(-t) - u_{k-1}(-t)) \\
&= (u_k(t) - u_{k-1}(t))(u_k(t) + u_{k-1}(t)) \\
&= (u_k^2 - u_{k-1}^2) \\
&= \left(\sum_{i=0}^{k} u_{2i} - \sum_{i=0}^{k-1} u_{2i}\right) \\
&= u_{2k} \\
&= -Q_{k+1}(t).
\end{aligned}$$

□

We leave the following example for the interested reader. Let $N[i,j] = (-1)^{i+j}\min(i,j)$ and $f_n = \det(I_n - tN)$. Then, $f_0 = 1$, $f_1 = 1-t$, $f_2 = t^2 - 3t + 1$, and $f_3 = -t^3 + 5t^2 - 6t + 1$. Then, $f_{n+1}f_{n-1} - f_n^2 = -t$, $A = t$, and $M = 2 - t$.

6. Tridiagonal Matrices

By expansion along the last row, the recurrence formula for the characteristic polynomial F_n of the tridiagonal matrix

$$Z_n = \begin{pmatrix}
a_1 & b_1 & 0 & \cdots & \\
c_1 & a_2 & b_2 & 0 & \cdots \\
\cdots & & & & \\
 & \cdots & & & \\
\cdots\cdots & 0 & c_{n-2} & a_{n-1} & b_{n-1} \\
\cdots & \cdots & 0 & c_{n-1} & a_n
\end{pmatrix}$$

is given by

$$F_n(t) = (a_n - t)F_{n-1}(t) - c_{n-1}b_{n-1}F_{n-2}(t).$$

Now, we assume $a_1 = s$, $a = a_i$ for $1 < i \leq n$, $b = b_i$, and $c_i = \frac{1}{b}$ for $i \leq n$. Then, $F_n(t) = (a-t)F_{n-1}(t) - F_{n-2}(t)$ for $n \geq 2$.

Corollary 6.1. *With the above assumptions, F_n is A-Cassini with $M = a - t$, $F_0 = 1$, and $F_1 = (s - t)$. Hence, $A = 1 + s^2 - sa + t(a - s)$.*

Proof. This follows immediately from Proposition 1.3 since $A = 1 + (s-t)^2 - (a-t)(s-t)$. □

In a similar manner, the next result can be obtained using Theorem 1.4.

Corollary 6.2. *With the above assumptions, $d_n = \det(Z_n)$ is A-Cassini with $A = 1 + s^2 - sa$, $M = a$, $d_0 = 1$, and $d_1 = s$. Hence, $d_n = su_n(a) - u_{n-1}(a)$ and is independent of the value of b.*

Suppose that Z_n is invertible. Then, $F_n = \det(Z_n - tI_n) = \det(Z_n)\det(I_n - tZ_n^{-1})$. There are well-known formulas for the inverse of Z_n, [7]. For $a = 2$, $s = 1$, and $b = c = -1$, we can use these formulas to see that $Z_n^{-1} = M_n$ from Section 5. However with $b = c = 1$ we obtain the matrix N from the example in Section 5. Also, using Corollary 1.9 with $s = 1$ and $a = 2$, $\det(Z_n) = 1$.

Additionally, if we fix the 1-1 entry and 2-2 entry to s_1 and s_2 and let the size n increase, we obtain similar recurrences for the tridiagonal matrices with a higher degree polynomial A. Also using the above ideas, if we let $bc = -1$, we get solutions to the Cassini relations discussed in [1]. We leave both of these for future investigations.

References

[1] R. C. Alperin, *A-Cassini sequences and their spectrum*, The Fibonacci Quarterly, **56.2** (2018), 153–155.

[2] R. C. Alperin, *Integer sequences generated by $x_{n+1} = (x_n^2 + A)/x_{n-1}$*, The Fibonacci Quarterly, **49.4** (2011), 362–365.

[3] M. Göbel, *Rewriting techniques and degree bounds for higher order symmetric polynomials*, AAECC, **9** (1999), 559–573.

[4] B. Junge and V. E. Hoggatt, Jr., *Polynomials arising from reflections across multiple plates*, The Fibonacci Quarterly, **11.3** (1973), 285–291.

[5] T. J. Rivlin, The Chebyshev Polynomials, Wiley-Interscience, New York, 1974.

[6] P. Steinbach, *Golden fields: A case for the heptagon*, Math. Mag., **70.1** (1997), 22–31.

[7] Wikipedia, `https:en.wikipedia.org/wiki/Tridiagonal_matrix`.

MSC2010: 11B37, 11B39, 11B83, 15B05

Department of Mathematics, San Jose State University, San Jose, CA 95192
E-mail address: rcalperin@gmail.com