EXPLICIT BOUNDS FOR THE DIOPHANTINE EQUATION $A!B! = C!$

LAURENT HABSIEGER

Abstract. A nontrivial solution of the equation $A!B! = C!$ is a triple of positive integers (A, B, C) with $A \leq B \leq C - 2$. It is conjectured that the only nontrivial solution is $(6, 7, 10)$, and this conjecture has been checked up to $C = 10^6$. Several estimates on the relative size of the parameters are known, such as the one given by Erdös, $C - B \leq 5 \log\log C$, or the one given by Bhat and Ramachandra, $C - B \leq (1/\log 2 + o(1)) \log\log C$. We check the conjecture for $B \leq 10^{3000}$ and give better explicit bounds such as $C - B \leq \frac{\log\log(B+1)}{\log 2} - 0.8803$.

1. Introduction

Many authors [6] considered the Diophantine equation

$$n! = \prod_{i=1}^{r} a_i! \tag{1.1}$$

in the integers $r, a_1, \ldots, a_r$, with $r \geq 2$ and $a_1 \geq \cdots \geq a_r \geq 2$. A trivial solution is given by $a_1 = n - 1$ and $n = \prod_{i=2}^{r} a_i!$. Hickerson conjectured that the only nontrivial solutions are $9! = 7!3!3!2!$, $10! = 7!6! = 7!5!3!$, and $16! = 14!5!2!$. He checked it for $n \leq 410$, which was improved to 18160 by Shallit and Easter (see [6]). Surányi also conjectured the case $r = 2$ (see [5]) and this was verified up to $n = 10^6$ by Caldwell [2].

Luca [8] proved there are finitely many nontrivial solutions to (1.1), assuming the *abc*-conjecture. Erdös [5] showed that, if the largest prime number of $n(n + 1)$ is greater than $4 \log n$ for any positive integer n, then there are only finitely many nontrivial solutions to (1.1).

From now on, we shall focus on the case $r = 2$, i.e., the equation

$$A!B! = C!\,, \tag{1.2}$$

which has been studied by Caldwell [2] for $C \leq 10^6$. Erdös [4] proved that $C - B \leq 5 \log\log C$ for C sufficiently large, and noted that it would be nice to obtain a bound of the form $C - B = o(\log\log C)$. His result was improved by Bhat and Ramachandra [1], who showed that $C - B \leq (1/\log 2 + o(1)) \log\log C$. Hajdu, Papp, and Szakács [7] recently proved that nontrivial solutions different from $10! = 7!6!$ satisfy $C < 5(B - A)$ and $B - A \geq 10^6$. The aim of this paper is to get better explicit inequalities.

Let $a \geq 2$ be an integer. Let s_a denote the sum of the digits of an integer written in the base a. When p is a prime, Legendre's formula gives the exponent of p in $n!$:

$$v_p(n!) = \frac{n - s_p(n)}{p - 1}\,.$$

When we apply this formula to (1.2), we find $A - v_p(A) + B - v_p(B) = C - v_p(C)$. Since $v_p(C) \geq 1$ and $v_p(n) \leq \frac{(p-1)\log(n+1)}{\log p}$ (see Lemma 1 below), we obtain

$$C \geq A + B + 1 - \frac{\log(A + 1)}{\log 2} - \frac{\log(B + 1)}{\log 2}\,. \tag{1.3}$$

Since $\log C! = \log A! + \log B!$, the condition (1.3) implies that A is much smaller that B. We shall make this assertion explicit by proving the following theorem.

Theorem 1.1. *Let $(A, B, C) \neq (6, 7, 10)$ be a nontrivial solutions triple of (1.2). For any real number $t > -1 - \frac{1+2\log\log 2}{\log 2} = -1.3851\ldots$, we have*

$$A \leq \frac{\log(B+1)}{\log 2} + \frac{2\log\log(B+1)}{\log 2} + t,$$

when B is sufficiently large. Moreover, we have

$$A \leq \frac{\log(B+1)}{\log 2} + \frac{2\log\log(B+1)}{\log 2} + 2.1221\,.$$

We can slightly improve on Bhat and Ramachandra's result [1].

Theorem 1.2. *Let $(A, B, C) \neq (6, 7, 10)$ be a nontrivial solution triple of (1.2). For any real number $u > -\frac{1+\log\log 2}{\log 2} = -0.9139\ldots$, we have*

$$C - B \leq \frac{\log\log(B+1)}{\log 2} + u,$$

when B is sufficiently large. Moreover, we have

$$C - B \leq \frac{\log\log(B+1)}{\log 2} + 1.819\,.$$

We also deduce a better explicit estimate than $B - A > C/5$ given by Hajdu, Papp, and Szakács [7].

Theorem 1.3. *Let $(A, B, C) \neq (6, 7, 10)$ be a nontrivial solution triple of (1.2). For any real number $v < 1 + \frac{2+3\log\log 2}{\log 2} = 2.299\ldots$, we have*

$$B - A > C - \frac{\log(C+1)}{\log 2} - \frac{3\log\log(C+1)}{\log 2} + v,$$

when B is sufficiently large. Moreover, we have

$$B - A > C - \frac{\log(C+1)}{\log 2} - \frac{3\log\log(C+1)}{\log 2} - 3.9411\,.$$

All these general estimates used $B \geq 10^6$ for nontrivial solutions triple distinct from $(6, 7, 10)$. We use these estimates to improve the range of validity of Surányi's conjecture and the estimates given before.

Theorem 1.4. *Let $(A, B, C) \neq (6, 7, 10)$ be a nontrivial solution triple of (1.2). Then, we have $B \geq 10^{3000}$ and*

$$A \leq \frac{\log(B+1)}{\log 2} + \frac{2\log\log(B+1)}{\log 2} - 1.3479\,,$$

$$C - B \leq \frac{\log\log(B+1)}{\log 2} - 0.8803\,,$$

$$B - A > C - \frac{\log(C+1)}{\log 2} - \frac{3\log\log(C+1)}{\log 2} + 2.2282\,.$$

Remark 1.5. *Theorem 1.4 extends Caldwell's result ($C \geq 10^6$) concerning the conjecture of Surányi to a much larger region ($C \geq 10^{3000}$).*

We first establish useful general properties for the sum of digits and for the Γ function in the next section. In Section 3, we prove a key lemma that studies the asymptotic behavior of $\log C! - \log A! - \log B!$ under the condition (1.3), for $A = \frac{\log(B+1)}{\log 2} + \frac{2\log\log(B+1)}{\log 2} + t$. We deduce Theorems 1.1–1.3 in Section 4. In Section 5, we use these results to prove Theorem 1.4, hence also to check Surányi's conjecture further, and to improve on the results of the preceding section. We end this paper with a few remarks on possible ways to get better results.

2. General Properties of s_a and Γ

We first give a tight upper bound for the sum of the digits function.

Lemma 2.1. *Let $a \geq 2$ be an integer. For any nonnegative integer n, we have the upper bound*

$$s_a(n) \leq \frac{(a-1)\log(n+1)}{\log a}\,.$$

Proof. Let n be a nonnegative integer. Write $s_a(n) = (a-1)b + r$, where b is a nonnegative integer and $0 \leq r \leq a-2$. We have

$$n \geq \sum_{i=0}^{b-1}(a-1)a^i + ra^b = (r+1)a^b - 1\,.$$

The function $x \to x - (a-1)\frac{\log(x+1)}{\log a}$ is convex and vanishes at $x = 0$ and $x = a-1$. Therefore, this function is nonpositive on the interval $[0, a-1]$. We thus get

$$s_a(n) = (a-1)b + r \leq (a-1)\frac{\log(a^b)}{\log a} + (a-1)\frac{\log(r+1)}{\log a} \leq \frac{(a-1)\log(n+1)}{\log a}\,.$$

□

Put $\Psi(z) = \Gamma'(z)/\Gamma(z)$. Let γ denote Euler's constant. We recall the formulas (see [3], p. 15)

$$\begin{aligned}\Psi(z) &= -\gamma + \sum_{k=0}^{\infty}\left(\frac{1}{k+1} - \frac{1}{z+k}\right)\\ \Psi'(z) &= \sum_{k=0}^{\infty}\frac{1}{(z+k)^2}\,,\end{aligned} \tag{2.1}$$

and Binet's second expression for $\log\Gamma$ (see [3], p. 22)

$$\log\Gamma(x) = \left(x - \frac{1}{2}\right)\log x - x + \frac{\log(2\pi)}{2} + 2\int_0^{\infty}\frac{\arctan(t/x)}{e^{2\pi t} - 1}dt\,. \tag{2.2}$$

From the bounds $0 \leq \arctan(t/x) \leq t/x$ and from (2.2), we get the well-known explicit Sirtling's formula

$$0 \leq \log\Gamma(x) - x(\log x - 1) - \frac{\log(2\pi/x)}{2} \leq \frac{1}{12x}\,. \tag{2.3}$$

Deriving (2.2) also leads to the formula

$$\Psi(x) = \log x - \frac{1}{2x} - \int_0^{\infty}\frac{2t}{(t^2+x^2)(e^{2\pi t}-1)}dt$$

and the bounds $0 \leq 1/(t^2+x^2) \leq 1/x^2$ give the estimates

$$-\frac{1}{12x^2} \leq \Psi(x) - \log x + \frac{1}{2x} \leq 0\,. \tag{2.4}$$

3. The Key Lemma

Let us define

$$R(A,B)=\log\Gamma\left(A+B+2-\frac{\log(A+1)}{\log 2}-\frac{\log(B+1)}{\log 2}\right)-\log\Gamma\left(A+1\right)-\log\Gamma\left(B+1\right),$$

and put

$$A_t=\frac{\log(B+1)}{\log 2}+\frac{2\log\log(B+1)}{\log 2}+t$$

for any real number t.

Lemma 3.1. *Let t be a real number such that $t>-1-\frac{1+2\log\log 2}{\log 2}=-1.3851\ldots$. There exists a function $C(t,B+1)$ such that*

$$R(A_t,B)\geq C(t,B+1)\log(B+1),$$

with

$$\lim_{B\to+\infty}C(t,B+1)=t+1+\frac{1+2\log\log 2}{\log 2}>0.$$

Moreover, we have $C(2.1221,B+1)>0$ for $B\geq 10^6$.

Proof. For $B\geq 2$, we have

$$\begin{aligned}\log\left(A_t+1\right)&=\log\left(\frac{\log(B+1)}{\log 2}+\frac{2\log\log(B+1)}{\log 2}+t+1\right)\\&\leq\log\log(B+1)-\log\log 2+\frac{2\log\log(B+1)+(t+1)\log 2}{\log(B+1)}\end{aligned}$$

and therefore,

$$\begin{aligned}A_t+B+2-\frac{\log(A_t+1)}{\log 2}-\frac{\log(B+1)}{\log 2}&=B+t+2+\frac{2\log\log(B+1)}{\log 2}-\frac{\log(A_t+1)}{\log 2}\\&\geq B+t+2+\frac{\log\log 2}{\log 2}+\frac{\log\log(B+1)}{\log 2}-\frac{2\log\log(B+1)+(t+1)\log 2}{\log 2\log(B+1)}>B+1\end{aligned}$$

for $B\geq 35$. We thus get, from (2.1) and (2.4) and for $B\geq 35$ that

$$\begin{aligned}&\log\Gamma\left(A_t+B+2-\frac{\log(A_t+1)}{\log 2}-\frac{\log(B+1)}{\log 2}\right)-\log\Gamma\left(B+1\right)\\&\quad\geq\left(\frac{\log\log(B+1)}{\log 2}+t+1+\frac{\log\log 2}{\log 2}-\frac{2\log\log(B+1)+(t+1)\log 2}{\log 2\log(B+1)}\right)\Psi(B+1)\\&\quad\geq\left(\frac{\log\log(B+1)}{\log 2}+t+1+\frac{\log\log 2}{\log 2}-\frac{2\log\log(B+1)+(t+1)\log 2}{\log 2\log(B+1)}\right)\\&\qquad\qquad\times\left(\log(B+1)-\frac{1}{2(B+1)}-\frac{1}{12(B+1)^2}\right).\\&\quad=\left(\frac{\log\log(B+1)}{\log 2}+t+1+\frac{\log\log 2}{\log 2}+\varphi_1(t,B+1)\right)\log(B+1)\end{aligned}$$

with

$$\begin{aligned}\varphi_1(t,x)=&-\frac{2\log\log x+(t+1)\log 2}{\log 2\log x}\\&-\frac{1}{\log x}\left(\frac{1}{2x}+\frac{1}{12x^2}\right)\left(\frac{\log\log x}{\log 2}+t+1+\frac{\log\log 2}{\log 2}-\frac{2\log\log x+(t+1)\log 2}{\log 2\log x}\right).\end{aligned}$$

Stirling's formula (2.3) gives

$$\log\Gamma(x) \leq x(\log x - 1) + \frac{\log(2\pi/x)}{2} + \frac{1}{12x} \leq x(\log x - 1)$$

for $x \geq 6.448$, from which we obtain

$$\log\Gamma\left(A_t + 1\right) \leq \left(\frac{\log(B+1)}{\log 2} + \frac{2\log\log(B+1)}{\log 2} + t + 1\right) \times \left(\log\log(B+1) - 1 - \log\log 2 + \frac{2\log\log(B+1) + (t+1)\log 2}{\log(B+1)}\right)$$

when $A_t \geq 6.448$. Since $A_t > A_{-1-\frac{1+2\log\log 2}{\log 2}} \geq 6.448$ for $B \geq 23$, we get

$$\log\Gamma\left(A_t + 1\right) \leq \left(\frac{\log\log(B+1)}{\log 2} - \frac{1+\log\log 2}{\log 2} + \varphi_2(t, B+1)\right)\log(B+1)$$

for $B \geq 23$, with

$$\varphi_2(t,x) = \frac{2\log\log x + (t+1)\log 2}{\log 2\log x}\left(\log\log x - \log\log 2 + \frac{2\log\log x + (t+1)\log 2}{\log x}\right).$$

We deduce

$$R(A_t, B) \geq \left(t + 1 + \frac{1+2\log\log 2}{\log 2} + \varphi_1(t, B+1) - \varphi_2(t, B+1)\right)\log(B+1)$$

for $B \geq 35$, and we put $C(t, B+1) = t + 1 + \frac{1+2\log\log 2}{\log 2} + \varphi_1(t, B+1) - \varphi_2(t, B+1)$. Note that the functions $\varphi_1(t,x)$ and $\varphi_2(t,x)$ tend to 0 when x goes to infinity, which proves the first part of the lemma.

For $t \geq -1$ and $x \geq 10^6$, we have

$$-C(t,x) \leq \frac{2\log\log x + (t+1)\log 2}{\log 2\log x}\left(\log\log x + 1 - \log\log 2 + \frac{2\log\log x + (t+1)\log 2}{\log x}\right) + \left(\frac{1}{2x} + \frac{1}{12x^2}\right)(0.2634 + 0.0672(t+1)) + t + 1 + \frac{1+2\log\log 2}{\log 2},$$

a decreasing function of x. We thus deduce $C(2.1221, 10^6) > 0.000016$, which completes the proof. □

4. Proof of the First Three Theorems

4.1. **Proof of Theorem 1.1.**

Lemma 4.1. *If (A,B,C) is a solution of* (1.2)*, then $R(A,B) \leq 0$. The function R is an increasing function of A for $1 \leq A \leq B$.*

Proof. The first claim follows directly from (1.3): $R(A,B) \leq \log C! - \log A! - \log B! = 0$. We compute

$$\frac{\partial^2 R}{\partial A \partial B}(A,B) = \left(1 - \frac{1}{(A+1)\log 2}\right)\left(1 - \frac{1}{(B+1)\log 2}\right)\Psi'\left(A + B + 2 - \frac{\log(A+1)}{\log 2} - \frac{\log(B+1)}{\log 2}\right).$$

From (2.1) we get $\frac{\partial^2 R}{\partial A \partial B}(A,B) \geq 0$ for $1 \leq A \leq B$. We use (2.1) to deduce

$$\begin{aligned}\frac{\partial R}{\partial A}(A,B) \geq \frac{\partial R}{\partial A}(A,A) &= \left(1 - \frac{1}{(A+1)\log 2}\right)\Psi\left(2A+2-2\frac{\log(A+1)}{\log 2}\right) - \Psi(A+1)\\ &= \frac{\gamma}{(A+1)\log 2} + \sum_{k=0}^{\infty}\left(\frac{1}{k+A+1} - \frac{1-\frac{1}{(A+1)\log 2}}{k+2A+2-2\frac{\log(A+1)}{\log 2}}\right)\\ &= \frac{\gamma}{(A+1)\log 2} + \sum_{k=0}^{\infty}\frac{\frac{k}{(A+1)\log 2} + A + 1 - 2\frac{\log(A+1)}{\log 2} + \frac{1}{\log 2}}{(k+A+1)(k+2A+2-2\frac{\log(A+1)}{\log 2})} > 0\end{aligned}$$

when $A+1 \geq \max\left(2\frac{\log(A+1)}{\log 2} - \frac{1}{\log 2}, \frac{\log(A+1)}{\log 2}\right) \geq 0$, which is true for $A \geq 1$. □

Thus, we only need to find $\bar{A}$ such that $R(\bar{A},B) > 0$ to get a bound $A < \bar{A}$. For $t > -1 - \frac{1+2\log\log 2}{\log 2}$, we have $R(A_t,B) > 0$ for B large enough by Lemma 3.1, which gives the first part of Theorem 1.1. Hajdu, Papp, and Szakács [7] proved $B - A \geq 10^6$, which ensures us that $B \geq 10^6$. We can therefore deduce the second part of the theorem from the inequality $C(2.1221, B+1) > 0$, also given in Lemma 3.1.

4.2. **Proof of Theorem 1.2.** Note that

$$\log A! = \log \frac{C!}{B!} \geq (C-B)\log(B+1).$$

For $A \leq A_t$, we have shown, in the proof of Lemma 3.1, that

$$\log A! \leq \log \Gamma(A_t+1) \leq \left(\frac{\log\log(B+1)}{\log 2} - \frac{1+\log\log 2}{\log 2} + \varphi_2(t,B+1)\right)\log(B+1).$$

Therefore,

$$C - B \leq \frac{\log\log(B+1)}{\log 2} - \frac{1+\log\log 2}{\log 2} + \varphi_2(t,B+1),$$

thus proving the first part of the theorem, since $\varphi_2(t,x)$ tend to 0 when x goes to infinity.

Each monomial term $(\log\log x)^n(\log x)^{-m}$ defining φ_2 is a positive decreasing function of x for $t \geq -1$ and $x \geq 10^6$. We find $-\frac{1+\log\log 2}{\log 2} + \varphi_2(2.1221, 10^6) < 1.819$ and the theorem follows, as in the previous subsection.

4.3. **Proof of Theorem 1.3.** We write $B - A = C - A - (C - B)$ and use Theorems 1.1 and 1.2 to get

$$B - A \geq C - \frac{\log(B+1)}{\log 2} - \frac{3\log\log(B+1)}{\log 2} - 3.9411.$$

The second part of the theorem follows, and the first part is straightforward.

5. The Proof of Theorem 1.4

Theorems 1.2 and 1.3 show that A and $C - B$ are small with respect to B. Let us put $k = C - B$ to simplify the statements.

Lemma 5.1. *Let (A,B,C) be a nontrivial solution triple of (1.2). For $k = C - B \in \{2,3,\ldots,20\}$, we have $B = B_k(A) = \lceil (A!)^{1/k} - (k+1)/2 \rceil$.*

Proof. We have

$$A! = \prod_{i=1}^{k}(B+i) = \prod_{i=1}^{k}\sqrt{(B+i)(B+k+1-i)} < \left(B + \frac{k+1}{2}\right)^k ,$$

which shows that $B > (A!)^{1/k} - (k+1)/2$.

We used MAPLE to check that the polynomial $\prod_{i=1}^{k}(B+i)-(B+(k-1)/2)^k$ is a polynomial in $B-1$ with nonnegative coefficients and with a positive value at $B=1$, for $2 \le k \le 12$. This implies that $B < (A!)^{1/k} - (k-1)/2$, and the lemma follows. □

We checked that the inequality $A! = \prod_{i=1}^{k}(B_k(A)+i)$ never occurred for $A \le 10000$ and $2 \le k \le 12$ using MAPLE; we asked for 40000-digit precision (enough to write all the digits of $A!$), and this required about 28 hours of computations.

For $B \le 10^{1000}$, Theorems 1.2 and 1.3 give $A \le 3346$ and $k \le 12$, so that the equation (1.2) has no nontrivial solution for $10^6 \le B \le 10^{1000}$. We can get better inequalities in these theorems, using $B \ge 10^{1000}$. Computing $C(-1.2979, 10^{1000})$ and $\varphi_2(1.2979, 10^{1000})$ leads to

$$A \le \frac{\log(B+1)}{\log 2} + \frac{2\log\log(B+1)}{\log 2} - 1.2979\,,$$

$$C - B \le \frac{\log\log(B+1)}{\log 2} - 0.8362\,.$$

For $10^{1000} \le B \le 10^{3000}$, we obtain $A \le 9993$ and $k \le 11$, and the equation (1.2) has no nontrivial solution for this interval. Computing $C(-1.3479, 10^{3000})$ and $\varphi_2(1.3479, 10^{3000})$ gives the inequalities from Theorem 1.4.

6. Concluding Remarks

Our method is based on two pieces of information: arithmetical information obtained by considering the dyadic valuation of the factorials, and asymptotic information obtained from Stirling's formula. To improve on the orders of magnitude of our estimates, one should get more arithmetical information. First, we applied the estimate from Lemma 2.1 for $A!$ and for $B!$, and it is uncommon that this estimate can be sharp in both cases. Second, we did not use any property of the p-adic valuations for $p \ge 3$, and any useful information could lead to improvements.

The algorithm we used to check that $A!B_k(A)! \ne (B_k(A)+k)!$ is basic. A smarter one should lead to an much larger bound than ours.

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Université de Lyon, CNRS UMR 5208, Université Claude Bernard Lyon 1, Institut Camille Jordan, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France

E-mail address: habsieger@math.univ-lyon1.fr