A CLASS OF EXPONENTIAL SEQUENCES WITH SHIFT-INVARIANT DISCRIMINATORS

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Abstract. The discriminator of an integer sequence $\mathbf{s} = (s(i))_{i\geq 0}$ with distinct terms, introduced by Arnold, Benkoski, and McCabe in 1985, is the function $D_{\mathbf{s}}(n)$ that sends n to the least integer m such that the n values $s(0)$, $s(1)$, $\ldots$, $s(n-1)$ are pairwise incongruent modulo m. In this note, we compute the discriminators for a class of exponential sequences that have the special property that the discriminator is *shift-invariant*, i.e., that it does not depend on the particular index the sequence is chosen to start with.

1. Discriminators

Let m be a positive integer. If S is a set of integers that are pairwise incongruent modulo m, we say that m *discriminates* S. Now let $\mathbf{s} = (s(i))_{i\geq 0}$ be a sequence of distinct integers. For all integers $n \geq 1$, we define $D_{\mathbf{s}}(n)$ to be the least positive integer m that discriminates the set $\{s(0),\ s(1),\ \ldots,\ s(n-1)\}$. The function $D_{\mathbf{s}}(n)$ is called the *discriminator* of the sequence $\mathbf{s}$.

The discriminator was first introduced by Arnold, Benkoski, and McCabe [1]. They derived the discriminator for the sequence 1, 4, 9, $\ldots$ of positive integer squares. More recently, discriminators of various sequences were studied by Schumer and Steinig [14], Barcau [2], Schumer [13], Bremser, Schumer, and Washington [3], Moree and Roskam [10], Moree [8], Moree and Mullen [9], Zieve [16], Sun [15], Moree and Zumalacárrequi [11], Ciolan and Moree [4], and the authors [5, 7].

In almost all of these cases, however, the discriminator is based on the first n terms of a sequence, for $n \geq 2$. Therefore, the discriminator can depend crucially on the starting point of a given sequence. For example, although the discriminator for the first three positive squares, $(1, 4, 9)$, is 6, we see that the number 6 does not discriminate the first length-3 "window" into the shifted sequence, $(4,\ 9,\ 16,\ \ldots)$, since $16 \equiv 4 \pmod 6$.

Furthermore, there has been little work on the discriminators of exponential sequences. Sun [15] presented some conjectures concerning certain exponential sequences, whereas Moree and Zumalacárrequi [11] computed the discriminator for the sequence $\left(\frac{|(-3)^j-5|}{4}\right)_{j\geq 0}$. The latter was generalized by Ciolan and Moree [4], who computed the discriminators for sequences of the form $(s_q(n))_{n\geq 1} = \left(\frac{3^n-q^*(-1)^n}{4}\right)_{n\geq 1}$, with $q^* = (-1)^{(q-1)/2}\cdot q$, an infinite family of sequences introduced by Jerzy Browkin.

We say that the discriminator of a sequence is *shift-invariant* if the discriminator for the sequence does not depend on the starting point of the sequence; that is, for all positive integers c, the discriminator of the sequence $(s(n))_{n\geq 1}$ is the same as the discriminator of the shifted sequence $(s(n{+}c))_{n\geq 0}$. This idea was briefly mentioned by Zieve [16], who considered sequences with discriminators that are shift-invariant for sufficiently large values of n.

In this paper, we present a class of exponential sequences $(\mathrm{ex}(n,t,a))_{n\geq 0}$ whose discriminators are shift-invariant for all $n \geq 1$. Here $\mathrm{ex}(n,t,a) := a^{\frac{t^{2n}-1}{2^b}}$, and $t \geq 3$ and a are odd integers, whereas b is the smallest positive integer such that $t \not\equiv \pm 1 \pmod{2^b}$. A typical example is the sequence $(\mathrm{ex}(n,3,1))_{n\geq 0} = \left(\frac{9^n-1}{8}\right)_{n\geq 0}$. We show that the discriminator for all sequences of this form is $D_{\mathrm{ex}}(n) = 2^{\lceil \log_2 n \rceil}$. Furthermore, we show that this discriminator is shift-invariant, i.e., it applies to every sequence $(\mathrm{ex}(n+c,t,a))_{n\geq 0}$ for $c \geq 0$. We define the shifted sequence $(\mathrm{exs}(n,t,a,c))_{n\geq 0}$ as follows:

$$\mathrm{exs}(n,t,a,c) := \mathrm{ex}(n+c,t,a) = a\frac{t^{2(n+c)}-1}{2^b}.$$

Specifically, our main result is the following theorem.

Theorem 1.1. *Let t, a, and c be integers such that $t \geq 3$ is odd, a is odd, $c \geq 0$, and let b be the smallest positive integer such that $t \not\equiv \pm 1 \pmod{2^b}$. Then the discriminator for the sequence $(\mathrm{exs}(n,t,a,c))_{n\geq 0}$ is*

$$D_{\mathrm{exs}}(n) = 2^{\lceil \log_2 n \rceil}.$$

The outline of the paper is as follows. We compute the discriminators for $(\mathrm{ex}(n,t,1))_{n\geq 0}$ and all of its shifts by providing matching upper and lower bounds. In Section 2, we obtain an upper bound. In Section 3, we prove a lemma relating to the lower bound. Finally, in Section 4, we put the results together to prove Theorem 1.1.

2. Upper Bound

In this section, we derive an upper bound for the discriminator of the sequence $(\mathrm{ex}(n,t,1))_{n\geq 0}$ and all of its shifts, given that $t \geq 3$ is an odd integer and b is the smallest positive integer such that $t \not\equiv \pm 1 \pmod{2^b}$.

The constraint on b is closely related to 2-adic valuations. For primes p and integers $c \neq 0$, the *p-adic* valuation of c, denoted by $\nu_p(c)$, is the unique integer e such that $p^e | c$ and $p^{e+1} \nmid c$. We use the following familiar properties of p-adic valuations (see, e.g., [12, p. 10]):

(1) $\nu_p(c_1c_2) = \nu_p(c_1) + \nu_p(c_2)$.
(2) If $c_1 \neq c_2$, $2 \nmid c_1c_2$, and $\nu_2(c_1 - c_2) = e \geq 2$, then $\nu_2(c_1^{2^r} - c_2^{2^r}) = e + r$ for every $r \geq 1$.

Lemma 2.1. *Let $t \geq 3$ be an odd integer, and let b be the smallest positive integer such that $t \not\equiv \pm 1 \pmod{2^b}$. Then for $k \geq 1$, the powers of t^2 form a cyclic subgroup of order 2^k in $(\mathbb{Z}/2^{k+b})^*$.*

Proof. The value of b is either $\nu_2(t-1)+1$ or $\nu_2(t+1)+1$, whichever is larger. Note that $t+1$ and $t-1$ are both even, and that $(t+1)-(t-1) = 2$, which means 4 does not divide both $t+1$ and $t-1$. Therefore, $\min(\nu_2(t+1), \nu_2(t-1)) = 1$ and so $b = \nu_2(t+1) + \nu_2(t-1)$. From the first property of p-adic valuations listed above, we have $b = \nu_2((t+1)(t-1)) = \nu_2(t^2-1)$. Note that this implies $2^{b+1} \nmid t^2 - 1$ and so $t^2 \not\equiv 1 \pmod{2^{b+1}}$.

From the second property listed above, setting $c_1 = t^2$ and $c_2 = 1$ implies $e = \nu_2(t^2-1) = b$, and so $\nu_2\left((t^2)^{2^r} - 1\right) = r + b$ for all $r \geq 1$. Therefore, for $r = k$, we have $(t^2)^{2^k} \equiv 1 \pmod{2^{k+b}}$. Note that for $r \geq 1$, we have $2^{r+b+1} \nmid (t^2)^{2^r} - 1$, so setting $r = k-1$ for $k \geq 2$ implies $(t^2)^{2^{k-1}} \not\equiv 1 \pmod{2^{k+b}}$. This is also true for $k = 1$, since $t^2 \not\equiv 1 \pmod{2^{b+1}}$.

In other words, for all $k \geq 1$, we have $(t^2)^{2^k} \equiv 1 \pmod{2^{k+b}}$ and $(t^2)^{2^{k-1}} \not\equiv 1 \pmod{2^{k+b}}$. It follows that the order of the subgroup generated by t^2 in $(\mathbb{Z}/2^{k+b})^*$ is 2^k. □

We use this lemma to establish an upper bound on the discriminator of $(\mathrm{ex}(n,t,1))_{n\geq 0}$ and all its shifts.

Lemma 2.2. *Let $t \geq 3$ be an odd integer, and let b be the smallest positive integer such that $t \not\equiv \pm 1 \pmod{2^b}$. Then for $k \geq 0$, the number 2^k discriminates every set of 2^k consecutive terms of the sequence* $(\mathrm{ex}(n,t,1))_{n\geq 0} = \left(\frac{t^{2n}-1}{2^b}\right)_{n\geq 0}$.

Proof. For every $i \geq 0$, it follows from Lemma 2.1 that the numbers

$$(t^2)^i,\ (t^2)^{i+1},\ \ldots,\ (t^2)^{i+2^k-1}$$

are distinct modulo 2^{k+b}. By subtracting 1 from every element, we see that the numbers

$$(t^2)^i - 1,\ (t^2)^{i+1} - 1,\ \ldots,\ (t^2)^{i+2^k-1} - 1$$

are distinct modulo 2^{k+b}. Furthermore, these numbers are also congruent to 0 modulo 2^b because $\nu_2(t^2-1) = b$. It follows that the set of quotients

$$\left\{\frac{(t^2)^i - 1}{2^b},\ \frac{(t^2)^{i+1} - 1}{2^b},\ \ldots,\ \frac{(t^2)^{i+2^k-1} - 1}{2^b}\right\}$$

consists of integers that are distinct modulo $\frac{2^{k+b}}{2^b} = 2^k$.

Such a set of quotients coincides with every set of 2^k consecutive terms of the sequence $(\mathrm{ex}(n,t,1))_{n\geq 0}$. Since the numbers in each set are distinct modulo 2^k, the desired result follows. $\square$

3. Lower Bound

In this section, we establish a lemma that is useful for proving the lower bound $D_{\mathrm{exs}}(n) \geq 2^{\lceil \log_2 n \rceil}$.

Lemma 3.1. *Let $t \geq 3$ be an odd integer, and let b be the smallest positive integer such that $t \not\equiv \pm 1 \pmod{2^b}$. Then for all $k \geq 0$ and $1 \leq m \leq 2^{k+1}$, there exists a pair of integers (i,j) such that $0 \leq i < j \leq 2^k$ and $t^{2i} \equiv t^{2j} \pmod{2^b m}$.*

Proof. Let the prime factorization of m be

$$m = 2^x \prod_{1\leq \ell \leq u} p_\ell^{y_\ell} \prod_{1 \leq \ell \leq v} q_\ell^{z_\ell},$$

where u, v, x, y_ℓ, $z_\ell \geq 0$, while p_1, p_2, $\ldots$, p_u are the prime factors of m that also divide t, and q_1, q_2, $\ldots$, q_v are the odd prime factors of m that do not divide t. For each $\ell \leq u$, let e_ℓ be the integer such that $p_\ell^{e_\ell} || t$, i.e., we have $p_\ell^{e_\ell} | t$ but $p_\ell^{e_\ell+1} \nmid t$.

We need to find a pair (i,j) such that $t^{2i} \equiv t^{2j} \pmod{2^b m}$. From the Chinese remainder theorem, we know it suffices to find a pair (i,j) such that

$$\begin{aligned} t^{2i} &\equiv t^{2j} \pmod{2^{x+b}}, \\ t^{2i} &\equiv t^{2j} \pmod{p_\ell^{y_\ell}},\ \text{for all } 1 \leq \ell \leq u, \\ \text{and } t^{2i} &\equiv t^{2j} \pmod{q_\ell^{z_\ell}},\ \text{for all } 1 \leq \ell \leq v. \end{aligned}$$

For the first of these equations, we know from Lemma 2.1 that $(t^2)^i \equiv (t^2)^{i+2^x} \pmod{2^{x+b}}$. In other words, it suffices to have $2^x | (j-i)$ to get $t^{2i} \equiv t^{2j} \pmod{2^{x+b}}$.

Next, we consider the u equations of the form $t^{2i} \equiv t^{2j} \pmod{p_\ell^{y_\ell}}$. Since $p_\ell^{e_\ell}$ is a factor of t, it follows that $(t^2)^{y_\ell/2e_\ell}$ is a multiple of $(p_\ell^{2e_\ell})^{y_\ell/2e_\ell} = p_\ell^{y_\ell}$. Therefore, $(t^2)^{y_\ell/2e_\ell} \equiv 0 \pmod{p_\ell^{y_\ell}}$. Any further multiplication by t^2 also yields 0 modulo $p_\ell^{y_\ell}$. Thus, it suffices to have $j > i \geq \frac{y_\ell}{2e_\ell}$ to ensure that $t^{2i} \equiv t^{2j} \pmod{p_\ell^{y_\ell}}$.

Finally, there are v equations of the form $t^{2i} \equiv t^{2j} \pmod{q_\ell^{z_\ell}}$. In each case, q_ℓ is co-prime to t, which means that $(t^2)^{\varphi(q_\ell^{z_\ell})/2} = t^{\varphi(q_\ell^{z_\ell})} \equiv 1 \pmod{q_\ell^{z_\ell}}$, where $\varphi(n)$ is Euler's totient function. As $\varphi(q_\ell^{z_\ell}) = q_\ell^{z_\ell-1}(q_\ell - 1)$, it is sufficient to have $\frac{q_\ell^{z_\ell-1}(q_\ell-1)}{2} | (j-i)$ to ensure that $t^{2i} \equiv t^{2j} \pmod{q_\ell^{z_\ell}}$.

Merging these ideas together, we choose the following values for i and j:

$$i = \max_{1\le\ell\le u} \left\lceil \frac{y_\ell}{2e_\ell} \right\rceil, \qquad j = \max_{1\le\ell\le u} \left\lceil \frac{y_\ell}{2e_\ell} \right\rceil + 2^x \prod_{1\le\ell\le v} \frac{q_\ell^{z_\ell-1}(q_\ell-1)}{2},$$

to ensure that $t^{2i} \equiv t^{2j} \pmod{2^b m}$. It is clear that $0 \le i < j$. To show that $j \le 2^k$, we first observe that

$$\begin{aligned} j &= \max_{1\le\ell\le u} \left\lceil \frac{y_\ell}{2e_\ell} \right\rceil + 2^x \prod_{1\le\ell\le v} \frac{q_\ell^{z_\ell-1}(q_\ell-1)}{2} = \max_{1\le\ell\le u} \left\lceil \frac{y_\ell}{2e_\ell} \right\rceil + \frac{2^x}{2^v} \prod_{1\le\ell\le v} q_\ell^{z_\ell-1}(q_\ell-1) \\ &\le \max_{1\le\ell\le u} \left\lceil \frac{y_\ell}{2} \right\rceil + \frac{2^x}{2^v} \prod_{1\le\ell\le v} q_\ell^{z_\ell} = \max_{1\le\ell\le u} \left\lceil \frac{y_\ell}{2} \right\rceil + \frac{m}{2^v \prod_{1\le\ell\le u} p_\ell^{y_\ell}}. \end{aligned}$$

We now consider the following two cases.

Case 1. $u = 0$**:** If $v = 0$ as well, then $j = 2^x = m < 2^{k+1}$, which means that $x \le k$ and thus $j \le 2^k$. Otherwise, if $v \ge 1$, then we have

$$j \le \max_{1\le\ell\le u} \left\lceil \frac{y_\ell}{2} \right\rceil + \frac{m}{2^v \prod_{1\le\ell\le u} p_\ell^{y_\ell}} = \frac{m}{2^v} \le \frac{m}{2} < \frac{2^{k+1}}{2} = 2^k.$$

Case 2. $u \ge 1$**:** Let r be such that $y_r = \max_{1\le\ell\le u} y_\ell$, and thus, p_r is the corresponding prime number with exponent y_r. Since $p_r^{y_r} \ge p_r \ge 3$, we have

$$j \le \max_{1\le\ell\le u} \left\lceil \frac{y_\ell}{2} \right\rceil + \frac{m}{2^v \prod_{1\le\ell\le u} p_\ell^{y_\ell}} \le \left\lceil \frac{y_r}{2} \right\rceil + \frac{m}{p_r^{y_r}} \le \frac{y_r}{2} + \frac{1}{2} + \frac{m}{3}.$$

Note that $y_r \le \log_{p_r} m \le \log_3 m \le \frac{m}{3}$ for positive integers m. We leave it to the reader to verify the last inequality. Thus,

$$j \le \frac{y_r}{2} + \frac{1}{2} + \frac{m}{3} \le \frac{m}{6} + \frac{1}{2} + \frac{m}{3} = \frac{m+1}{2}.$$

Since m and j are integers, this implies that

$$j \le \left\lceil \frac{m}{2} \right\rceil \le \left\lceil \frac{2^{k+1}}{2} \right\rceil \le 2^k.$$

In both cases, we have $j \le 2^k$, thus fulfilling the required conditions. □

4. Discriminator of $(\mathrm{ex}(n,t,a))_{n\ge0}$ and Its Shifted Counterparts

In this section, we combine the results of the previous sections to determine the discriminator for $(\mathrm{ex}(n,t,a))_{n\ge1}$, as well as its shifted counterparts. We first prove a general lemma about the discriminator of some scaled sequences.

Lemma 4.1. *Given a sequence $s(0)$, $s(1)$, ..., and a nonzero integer a, let $s'(0)$, $s'(1)$, ..., denote the sequence such that $s'(i) = a \cdot s(i)$ for all $i \ge 0$. Then, for every n such that* $\gcd(|a|, D_s(n)) = 1$, *we have $D_{s'}(n) = D_s(n)$.*

Proof. From the definition of the discriminator, we know that for every $m < D_s(n)$, there exists a pair of integers i and j with $i < j < n$, such that $m|s(j) - s(i)$. Thus, for this same pair of i and j, we have

$$m|a(s(j) - s(i)) = as(j) - as(i) = s'(j) - s'(i).$$

Therefore, m cannot discriminate the set $\{s'(0),\ s'(1),\ \ldots,\ s'(n-1)\}$ and so $D_{s'}(n) \geq D_s(n)$.

But for $m = D_s(n)$, we know that for all i and j with $i < j < n$, we have $m \nmid s(j) - s(i)$. Since $\gcd(m, |a|) = 1$, it follows that

$$m \nmid a(s(j) - s(i)) = as(j) - as(i) = s'(j) - s'(i)$$

for all i and j with $i < j < n$. Therefore, $m = D_s(n)$ discriminates the set

$$\{s'(0),\ s'(1),\ \ldots,\ s'(n-1)\}$$

and so $D_{s'}(n) \leq D_s(n)$.

Putting these results together, we have $D_{s'}(n) = D_s(n)$. □

We can now prove Theorem 1.1.

Proof. First we compute the discriminator for $a = 1$, where the sequence is of the form $(\mathrm{exs}(n, t, 1, c))_{n\geq 0} = \left(\frac{(t^2)^{n+c}-1}{2^b}\right)_{n\geq 0}$.

The case for $n = 1$ is trivial. Otherwise, let $k \geq 0$ be such that $2^k < n \leq 2^{k+1}$. We show that $D_{\mathrm{exs}}(n) = 2^{k+1}$.

From Lemma 2.2, we know that 2^{k+1} discriminates the set,

$$\{\mathrm{ex}(c, t, 1),\ \mathrm{ex}(c + 1, t, 1),\ \ldots,\ \mathrm{ex}(c + 2^{k+1} - 1, t, 1)\},$$

as well as every smaller subset of these numbers. Therefore, 2^{k+1} discriminates

$$\{\mathrm{exs}(0, t, 1, c),\ \mathrm{exs}(1, t, 1, c),\ \ldots,\ \mathrm{exs}(n - 1, t, 1, c)\}.$$

In other words, $D_{\mathrm{exs}}(n) \leq 2^{k+1}$ for $a = 1$.

Now, let m be a positive integer such that $m < 2^{k+1}$. By Lemma 3.1, we know that there exists a pair of integers, i and j, such that $(t^2)^i \equiv (t^2)^j \pmod{2^b m}$. This implies that $(t^2)^{i+c} - 1 \equiv (t^2)^{j+c} - 1 \pmod{2^b m}$.

Furthermore, since $\nu_2(t^2 - 1) = b$, we have $(t^2)^{i+c} - 1 \equiv (t^2)^{j+c} - 1 \equiv 1 - 1 \equiv 0 \pmod{2^b}$. Therefore,

$$\frac{(t^2)^{i+c} - 1}{2^b} \equiv \frac{(t^2)^{j+c} - 1}{2^b} \pmod{m}.$$

In other words, $\mathrm{exs}(i, t, 1, c) \equiv \mathrm{exs}(j, t, 1, c) \pmod{m}$ while both numbers are in the set

$$\{\mathrm{exs}(0, t, 1, c),\ \mathrm{exs}(1, t, 1, c),\ \ldots,\ \mathrm{exs}(n - 1, t, 1, c)\}$$

since $i < j \leq 2^k < n$. Therefore, m fails to discriminate this set. Since this applies for all $m < 2^{k+1}$, we have $D_{\mathrm{exs}}(n) \geq 2^{k+1}$ for $a = 1$.

Since we have $2^{k+1} \leq D_{\mathrm{exs}} \leq 2^{k+1}$, this means that $D_{\mathrm{exs}}(n) = 2^{k+1}$ and thus $D_{\mathrm{exs}}(n) = 2^{\lceil \log_2 n \rceil}$, provided that $a = 1$.

As for $a \neq 1$, we observe that the value of $2^{\lceil \log_2 n \rceil}$ is a power of 2 for all n, and so it is co-prime to all odd a. Therefore, we can apply Lemma 4.1 to prove that the discriminator remains unchanged for odd values of a, thus proving that the discriminator for the sequence $(\mathrm{exs}(n, t, a, c))_{n\geq 0} = \left(a\frac{(t^2)^{n+c}-1}{2^b}\right)_{n\geq 0}$ is $D_{\mathrm{exs}}(n) = 2^{\lceil \log_2 n \rceil}$. □

5. Final Remarks

We have considered sequences of the form $(\mathrm{ex}(n,t,a))_{n\geq 0} = \left(a^{\frac{t^{2n}-1}{2^b}}\right)_{n\geq 0}$ for odd integers a and t, where b is the smallest positive integer such that $t \not\equiv \pm 1 \pmod{2^b}$. We showed that the discriminator for this sequence is characterized by $D_{\mathrm{ex}}(n) = 2^{\lceil \log_2 n \rceil}$ and that the discriminator is shift-invariant, i.e., all sequences of the form $(\mathrm{ex}(n+c,t,a))_{n\geq 0}$ for $c \geq 0$ share the same discriminator.

This raises the obvious question, what other sequences have shift-invariant discriminators? It is easy to show that sequences defined by a linear equation, i.e., of the form $(an+b)_{n\geq 0}$, have shift-invariant discriminators. Furthermore, the first author has recently shown [6] that the quadratic sequence $(2^k cn^2 + bcn)_{n\geq 0}$, for a positive integer k and odd integers b, c, also has a shift-invariant discriminator $2^{\lceil \log_2 n \rceil}$.

It is an open problem as to whether there are any sequences, other than those mentioned here, whose discriminators are shift-invariant. Furthermore, all sequences whose discriminators are known to be shift-invariant have discriminators with linear growth, but we do not know if this is true of all shift-invariant discriminators.

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