FAULHABER AND BERNOULLI

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Abstract. In this note, we will use Faulhaber's Formula to explain why the odd Bernoulli numbers are equal to zero.

1. Introduction

For odd numbers greater than or equal to seven, why are the Bernoulli numbers equal to zero? Because Faulhaber's Formula tells us that $\sum_{k=1}^{n} k^{2m+1}$ is a polynomial in $(\sum_{k=1}^{n} k)^2$, and $(\sum_{k=1}^{n} k)^2 = \frac{n^2+2n^3+n^4}{4}$.

2. Faulhaber's Formula

We might already know that

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{k=1}^{n} k\right)^2.$$

Through inductive reasoning like that in [6], we might discover further that

$$\begin{aligned}
2^1 \cdot \left(\frac{n(n+1)}{2}\right)^2 &= \binom{2}{1} \cdot \sum k^3 \\
2^2 \cdot \left(\frac{n(n+1)}{2}\right)^3 &= \binom{3}{0} \cdot \sum k^3 + \binom{3}{2} \cdot \sum k^5 \\
2^3 \cdot \left(\frac{n(n+1)}{2}\right)^4 &= \binom{4}{1} \cdot \sum k^5 + \binom{4}{3} \cdot \sum k^7 \qquad (2.1)\\
2^4 \cdot \left(\frac{n(n+1)}{2}\right)^5 &= \binom{5}{0} \cdot \sum k^5 + \binom{5}{2} \cdot \sum k^7 + \binom{5}{4} \cdot \sum k^9 \\
2^5 \cdot \left(\frac{n(n+1)}{2}\right)^6 &= \binom{6}{1} \cdot \sum k^7 + \binom{6}{3} \cdot \sum k^9 + \binom{6}{5} \cdot \sum k^{11}.
\end{aligned}$$

(We abbreviate $\sum_{k=1}^{n} k^m$ by $\sum k^m$.) The general case is

$$\begin{aligned}
2^{m-1} \cdot \left(\frac{n(n+1)}{2}\right)^m &= \binom{m}{0} \cdot \sum k^m + \binom{m}{2} \cdot \sum k^{m+2} \qquad (2.2)\\
&\quad + \cdots + \binom{m}{m-3} \cdot \sum k^{2m-3} + \binom{m}{m-1} \cdot \sum k^{2m-1}
\end{aligned}$$

or

$$\begin{aligned}
2^{m-1} \cdot \left(\frac{n(n+1)}{2}\right)^m &= \binom{m}{1} \cdot \sum k^{m+1} + \binom{m}{3} \cdot \sum k^{m+3} \qquad (2.3)\\
&\quad + \cdots + \binom{m}{m-3} \cdot \sum k^{2m-3} + \binom{m}{m-1} \cdot \sum k^{2m-1}.
\end{aligned}$$

If we wish to prove such expressions, by [1, 2, 3] we proceed using Pascal's observation of telescoping sums. Consider the special case of

$$\begin{aligned}\left(\frac{3\cdot 4}{2}\right)^m &= \left(\frac{1\cdot 2}{2}\right)^m - 0 + \left(\frac{2\cdot 3}{2}\right)^m - \left(\frac{1\cdot 2}{2}\right)^m + \left(\frac{3\cdot 4}{2}\right)^m - \left(\frac{2\cdot 3}{2}\right)^m \\ &= \sum_{k=1}^{3}\left[\left(\frac{k(k+1)}{2}\right)^m - \left(\frac{(k-1)k}{2}\right)^m\right].\end{aligned}$$

For the general case of

$$\left(\frac{n(n+1)}{2}\right)^m = \sum_{k=1}^{n}\left[\left(\frac{k(k+1)}{2}\right)^m - \left(\frac{(k-1)k}{2}\right)^m\right], \tag{2.4}$$

we rewrite the expression in brackets as

$$\left(\frac{k}{2}\right)^m ((1+k)^m - (-1+k)^m)$$

and then use the binomial theorem to arrive at (2.2) or (2.3).

What if we want to rewrite such expressions in terms of a particular $\sum k^{2m+1}$? For example, by looking at (2.1) we write

$$\begin{aligned}\sum k^5 &= \frac{1}{\binom{3}{2}}\cdot\left[2^2\cdot\left(\frac{n(n+1)}{2}\right)^3 - \binom{3}{0}\cdot\sum k^3\right] \\ &= \frac{1}{3}\cdot\left[4\cdot\frac{n(n+1)}{2} - 1\right]\cdot\left(\sum k\right)^2,\end{aligned}$$

which implies

$$\begin{aligned}\sum k^7 &= \frac{1}{\binom{4}{3}}\cdot\left[2^3\cdot\left(\frac{n(n+1)}{2}\right)^4 - \binom{4}{1}\cdot\sum k^5\right] \\ &= \frac{1}{4}\cdot\left[2^3\cdot\left(\frac{n(n+1)}{2}\right)^2 - 4\cdot\frac{1}{3}\cdot\left(4\cdot\frac{n(n+1)}{2} - 1\right)\right]\cdot\left(\sum k\right)^2.\end{aligned}$$

By a proof by mathematical induction on the m of (2.2) or (2.3), we arrive at the general result of

$$\begin{aligned}\sum k^{2m+1} &= \frac{1}{m+1}\cdot\left[2^m\cdot\left(\frac{n(n+1)}{2}\right)^{m-1} - a_2\cdot\left(\frac{n(n+1)}{2}\right)^{m-2} + a_3\cdot\left(\frac{n(n+1)}{2}\right)^{m-3}\right. \\ &\left.\mp\cdots\mp a_{m-2}\cdot\left(\frac{n(n+1)}{2}\right)^2 \mp a_{m-1}\cdot\frac{1}{3}\cdot\left(4\cdot\frac{n(n+1)}{2} - 1\right)\right]\cdot\left(\sum k\right)^2, \end{aligned}\tag{2.5}$$

where the a_i are rational numbers and $m \geqq 3$. We call this relationship Faulhaber's Formula. (For some of the history of the problem, see [1, 2, 3, 4].)

3. Bernoulli Numbers

By [1, 2] or Chapter 1 of [5], we define the Bernoulli numbers B_n by

$$B_0 = 1, \quad \sum_{k=0}^{n}\binom{n+1}{k}\cdot B_k = 0,$$

where $n \geqq 1$. For example, to find B_1 we write

$$\sum_{k=0}^{1} \binom{1+1}{k} \cdot B_k = \binom{2}{0} \cdot B_0 + \binom{2}{1} \cdot B_1 = 0,$$

which implies $B_1 = -\frac{1}{2}$. It turns out $B_3 = B_5 = 0$. Following the section "Back to Faulhaber's form" of [4] we write

$$\begin{aligned} \sum k^{2m+1} &= \frac{1}{2m+2} \cdot \left[\binom{2m+2}{0} \cdot B_0 \cdot n^{2m+2} - \binom{2m+2}{1} \cdot B_1 \cdot n^{2m+1} \right. \\ &\left. \pm \cdots + \binom{2m+2}{2m} \cdot B_{2m} \cdot n^2 - \binom{2m+2}{2m+1} \cdot B_{2m+1} \cdot n \right], \end{aligned} \tag{3.1}$$

for which we will assume $m \geqq 3$.

4. Conclusion

With regard to the claim at the start, suppose we set (2.5) and (3.1) equal to one another. If we multiply out (2.5), we see it does not contain the term n. That means the last term of (3.1),

$$-\frac{1}{2m+2} \cdot \binom{2m+2}{2m+1} \cdot B_{2m+1} \cdot n,$$

must be equal to zero. In other words, for all $m \geqq 3$, $B_{2m+1} = 0$.

Acknowledgement

The author thanks the anonymous referee for suggestions that improved the quality of the paper.

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MSC2010: 05A10, 11B68

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