# LINEAR RECURRING SEQUENCE SUBGROUPS IN THE COMPLEX FIELD - II

OWEN J. BRISON AND J. EURICO NOGUEIRA

Abstract. In [1], the authors studied $f$-subgroups in the complex field for polynomials $f(t)$ of degree 2 and described their behavior in many cases. Here, similar questions are considered for polynomials of general degree $k \geq 2$.

## 1. Introduction

Let $\mathbb{C}$ denote the complex field, let $\mathbb{C}_0[t]$ denote the set of nonconstant monic polynomials over $\mathbb{C}$ with nonzero constant coefficient, and let $\mathbb{C}^*$ denote the multiplicative group of nonzero complex numbers. Let $f(t) = t^k - a_{k-1}t^{k-1} - \cdots - a_1 t - a_0 \in \mathbb{C}_0[t]$. An *$f$-sequence* in $\mathbb{C}$ is a (doubly-infinite) sequence $\mathcal{S} = (s_i)_{i\in\mathbb{Z}}$ of elements $s_i \in \mathbb{C}$ such that

$$s_i = a_{k-1}s_{i-1} + \cdots + a_1 s_{i-k+1} + a_0 s_{i-k}$$

for all $i \in \mathbb{Z}$; $\mathcal{S}$ is *cyclic* if there exists $g \in \mathbb{C}^*$ such that $s_{i+1} = gs_i$ for all $i \in \mathbb{Z}$, in which case $g$ is the *common ratio* of $\mathcal{S}$; $\mathcal{S}$ is *periodic* if there exists $m \in \mathbb{N}$ such that $s_a = s_{a+m}$ for all $a \in \mathbb{Z}$, whereas the least such $m$ is called the *least period* of $\mathcal{S}$. By a *minimal periodic segment*, we understand the whole sequence, if $\mathcal{S}$ is not periodic, and any segment consisting of $m$ consecutive members of $\mathcal{S}$, if $\mathcal{S}$ is periodic with least period $m$.

If $g \in \mathbb{C}^*$ is a root of $f(t) \in \mathbb{C}_0[t]$, then the subgroup $< g > \leq \mathbb{C}^*$ may be regarded as (the underlying set of) an $f$-sequence:

$$< g > = (\ldots, 1, g, g^2, \ldots, g^n, \ldots).$$

It sometimes happens, for certain choices of $f(t)$ and $g$ with $f(g) = 0$, that $< g >$ may be written in an alternative manner as an $f$-sequence. This raises the question of classifying this behavior. An analogous situation for finite fields seems to have first been investigated by Somer [3, 4]. In [1], the authors studied this problem for $f(t) \in \mathbb{C}_0[t]$ of degree 2. In that paper, the case where $< g >$ is finite was described, whereas the case where $< g >$ is infinite was described except for certain specified situations: see Propositions 4 and 6 of [1].

**Definition 1.1.** *Let $f(t) \in \mathbb{C}_0[t]$. The subgroup $M \leq \mathbb{C}^*$ is said to be an $f$-subgroup if either*

*(i) $M$ is infinite and the underlying set of $M$ can be written in such a way as to form an $f$-sequence $(s_n)_{n\in\mathbb{Z}}$, where $s_a \neq s_b$ if $a \neq b$, or*

The first author acknowledges the partial support of the *Fundação para a Ciência e Tecnologia* via the project UID/MAT/04721/2013 (*Centro de Análise Funcional, Estruturas Lineares e Aplicações - Grupo de Estruturas Lineares, Algébricas e Combinatórias*, of the Universidade de Lisboa, Portugal). The second author acknowledges the partial support of the *Fundação para a Ciência e Tecnologia* via the project UID/MAT/00297/2019 (*Centro de Matemática e Aplicações*, of the Universidade Nova de Lisboa, Portugal).

*(ii) $M$ is finite, of order $m$, and the underlying set of $M$ can be written in such a way as to coincide with a minimal periodic segment of an $f$-sequence $(s_n)_{n\in\mathbb{Z}}$, where $s_a = s_b$ if and only if $a \equiv b \pmod{m}$.*

When $M$ is an $f$-subgroup, we will write $M = (s_n)_{n\in\mathbb{Z}}$, even if $M$ is finite, and say $(s_n)_{n\in\mathbb{Z}}$ is *a representation of*, or *represents*, the subgroup $M$ as an $f$-sequence.

The following lemma relates $f$-subgroups with cyclic $f$-sequences.

**Lemma 1.2.** *Let $f(t) = t^k - a_{k-1}t^{k-1} - \cdots - a_1t - a_0 \in \mathbb{C}_0[t]$.*

*(a) If $f(g) = 0$, then $(g^n)_{n\in\mathbb{Z}}$ is a cyclic $f$-sequence that represents $< g >$.*

*(b) Suppose $\mathcal{S} \subseteq \mathbb{C}^*$ is a cyclic $f$-sequence with common ratio $g$ and contains 1. Then, $f(g) = 0$ and $\mathcal{S}$ represents $< g > \leq \mathbb{C}^*$ as an $f$-subgroup.*

*Proof.* (a) We observed this above.

(b) Write $\mathcal{S} = (s_n)_{n\in\mathbb{Z}}$ and assume, without loss of generality, that $s_0 = 1$; then $s_n = g^n$ for all $n \in \mathbb{Z}$ and $\mathcal{S} = (g^n)_{n\in\mathbb{Z}}$ represents $< g > \leq \mathbb{C}^*$ as an $f$-subgroup. Because $\mathcal{S}$ is an $f$-sequence, $g^k = a_{k-1}g^{k-1} + \cdots + a_1g + a_0$ and so, $f(g) = 0$. □

If $f(t) \in \mathbb{C}_0[t]$, and if $g, h \in \mathbb{C}^*$ are distinct roots of $f$, it can happen that $< h > = < g > \leq \mathbb{C}^*$, and then, $(g^n)_{n\in\mathbb{Z}}$ and $(h^n)_{n\in\mathbb{Z}}$ are both "obvious", and cyclic, representations of $< g >$ as an $f$-subgroup. This suggests:

**Definition 1.3.** *Let $f(t) \in \mathbb{C}_0[t]$ be of degree $k \geq 2$.*

*(a) The $f$-subgroup $M$ of $\mathbb{C}^*$ is said to be* standard *if whenever $M$ is represented as an $f$-sequence $M = (s_n)_{n\in\mathbb{Z}}$, then $(s_n)_{n\in\mathbb{Z}}$ is necessarily cyclic. Otherwise, $M$ is said to be* nonstandard.

*(b) Suppose that $M$ is a nonstandard $f$-subgroup. If $M$ admits a representation as a cyclic $f$-sequence, then $M$ is* nonstandard of the first type; *otherwise $M$ is* nonstandard of the second type.

It is possible to find polynomials $f(t) \in \mathbb{C}_0[t]$ that admit noncyclic $f$-subgroups and, are thus nonstandard of the second type: see Theorem 3.3 below and Example 7 of [1]. The classification of these noncyclic $f$-subgroups seems to be an open problem; the only ones we have found are of the form $< a > \times < b >$, where $< a >$ is finite and $< b >$ is infinite.

Suppose $f(t) = t^k - a_{k-1}t^{k-1} - \cdots - a_0 \in \mathbb{C}_0[t]$ and $(s_i)_{i\in\mathbb{Z}}$ is an $f$-sequence in $\mathbb{C}$. Write $f^*(t) = t^k + \frac{a_1}{a_0}t^{k-1} + \cdots + \frac{a_{k-1}}{a_0}t - \frac{1}{a_0} \in \mathbb{C}_0[t]$; this is the monic version of the *reciprocal polynomial* of $f(t)$ and its roots are the inverses of the roots of $f(t)$. It is clear that $f^{**} = f$. For $i \in \mathbb{Z}$, write $u_i = s_{-i}$. Then, $(u_i)_{i\in\mathbb{Z}}$ (which is just "$(s_i)$ backwards") is an $f^*$-sequence and $M$ is an $f$-subgroup if and only if it is an $f^*$-subgroup.

## 2. The Main Result

The following result generalizes part (1) of Proposition 4 of [1].

**Theorem 2.1.** *Let $f(t) \in \mathbb{C}_0[t]$ be of degree $k \geq 2$ with roots $g_1, \ldots, g_k \in \mathbb{C}^*$ such that $|g_i| < |g_j|$ if $i < j$, and let $M$ be an $f$-subgroup. Then, $M$ is standard. Furthermore, $M$ is finite if and only if for some $i \in \{1, \ldots, k\}$ we have $M = < g_i >$, where $g_i$ is a root of unity.*

*Proof.* Let $(s_n)_{n\in\mathbb{Z}}$ be an $f$-sequence that represents $M$; then there exist $\alpha_i \in \mathbb{C}$ such that

$$s_n = \alpha_1 g_1^n + \alpha_2 g_2^n + \cdots + \alpha_k g_k^n$$

for all $n \in \mathbb{Z}$. Without loss of generality, assume that $s_0 = 1$. We wish to prove that precisely one of the $\alpha_i$ is nonzero. Thus, for a contradiction, suppose at least two of the $\alpha_i$ are nonzero. Let

$$k_2 = \min\{i : i \in \{1,\dots,k\}, \alpha_i \neq 0\} \quad \text{and} \quad k_1 = \max\{i : i \in \{1,\dots,k\}, \alpha_i \neq 0\},$$

so that $k_2 < k_1$.

Write $R_i = g_i/g_{k_1}$ for $i = k_2,\dots,k_1-1$ and $P_j = g_j/g_{k_2}$ for $j = k_2+1,\dots,k_1$. For all relevant $i$ and $j$, we have $|R_i| < 1$, $|P_j| > 1$, and

$$\lim_{n\to\infty} R_i^n = \lim_{n\to-\infty} P_j^n = 0. \tag{2.1}$$

We also have

$$\begin{aligned} |s_n| &= |g_{k_1}|^n |\alpha_{k_2} R_{k_2}^n + \ldots + \alpha_{k_1-1} R_{k_1-1}^n + \alpha_{k_1}| \\ &= |g_{k_2}|^n |\alpha_{k_2} + \alpha_{k_2+1} P_{k_2+1}^n + \ldots + \alpha_{k_1} P_{k_1}^n|. \end{aligned} \tag{2.2}$$

Thus,

$$\begin{aligned} \lim_{n\to\infty} |s_n| &= |\alpha_{k_1}| \lim_{n\to\infty} |g_{k_1}|^n, \\ \lim_{n\to-\infty} |s_n| &= |\alpha_{k_2}| \lim_{n\to-\infty} |g_{k_2}|^n, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \frac{|s_{n+1}|}{|s_n|} &= |g_{k_1}| \frac{|\alpha_{k_2} R_{k_2}^{n+1} + \cdots + \alpha_{k_1}|}{|\alpha_{k_2} R_{k_2}^n + \cdots + \alpha_{k_1}|} \\ &= |g_{k_2}| \frac{|\alpha_{k_2} + \cdots + \alpha_{k_1} P_{k_1}^{n+1}|}{|\alpha_{k_2} + \cdots + \alpha_{k_1} P_{k_1}^n|} \end{aligned} \tag{2.4}$$

so that

$$\begin{aligned} \lim_{n\to\infty} \frac{|s_{n+1}|}{|s_n|} &= |g_{k_1}| > 0, \\ \lim_{n\to-\infty} \frac{|s_{n+1}|}{|s_n|} &= |g_{k_2}| > 0. \end{aligned} \tag{2.5}$$

The following four assertions follow from (2.3) and (2.5).

(a) If $|g_{k_1}| > 1$, then there exists $n_0 \in \mathbb{N}$ such that $|s_{n+1}| > |s_n|$ for $n > n_0$ and $\lim_{n\to\infty} |s_n| = \infty$.

(b) If $|g_{k_2}| > 1$, then there exists $n_1 \in \mathbb{N}$ such that $|s_{n+1}| > |s_n|$ for $n < -n_1$ and $\lim_{n\to-\infty} |s_n| = 0$.

(c) If $|g_{k_1}| < 1$, then there exists $n_2 \in \mathbb{N}$ such that $|s_{n+1}| < |s_n|$ for $n > n_2$ and $\lim_{n\to\infty} |s_n| = 0$.

(d) If $|g_{k_2}| < 1$, then there exists $n_3 \in \mathbb{N}$ such that $|s_{n+1}| < |s_n|$ for $n < -n_3$ and $\lim_{n\to-\infty} |s_n| = \infty$.

The following two assertions follow from (2.3).

(e) If $|g_{k_1}| = 1$, then $\lim_{n\to\infty} |s_n| = |\alpha_{k_1}| > 0$.

(f) If $|g_{k_2}| = 1$, then $\lim_{n\to-\infty} |s_n| = |\alpha_{k_2}| > 0$.

There are now a number of cases to consider; the first three lead rapidly to contradictions.

Case (1). First, suppose $|g_{k_2}| < 1 < |g_{k_1}|$. By (a) and (d), there exists $n_4 \in \mathbb{N}$ such that $|s_n| > 1$ whenever $|n| \geq n_4$; in particular, the group $M$ contains a finite number of elements $s_n$ with $|s_n| < 1$, but an infinite number with $|s_n| > 1$, which is a contradiction because $M$ contains the inverse of each of its elements.

Case (2). Next, suppose $|g_{k_2}| = 1 < |g_{k_1}|$. By (a), $M$ contains elements of arbitrarily large modulus. By (a) and (f), the elements of $M$ are, in modulus, bounded below away from 0. This is again a contradiction.

Case (3). Next, suppose $|g_{k_2}| < 1 = |g_{k_1}|$. By (d), $M$ contains elements of arbitrarily large modulus. By (d) and (e), the elements of $M$ are, in modulus, bounded below away from 0. This is again a contradiction.

Case (4). Next, suppose $1 < |g_{k_2}| < |g_{k_1}|$. By (a) and (b), there exists $n_5 \in \mathbb{N}$ such that $|s_n|$ is monotonic increasing whenever $|n| \geq n_5$ and such that, $|s_{-n_5}| < |s_{n_5}|$. We also have $\lim_{n\to\infty} |s_n| = \infty$ and $\lim_{n\to-\infty} |s_n| = 0$. We start by proving that, in this case, $|s_u| \neq |s_v|$, whenever $u \neq v$. Write $N_5 = \{-n_5, \ldots, n_5\}$; then $|s_i|$ is monotonic increasing on $\mathbb{N}\backslash N_5$. In particular, $|s_u| \neq |s_v|$ whenever $u$ and $v$ are distinct elements of $\mathbb{N}\backslash N_5$. Suppose that $u \in N_5$ and $v \in \mathbb{N}$ are distinct with $|s_u| = |s_v|$; because $u \neq v$, then $s_u \neq s_v$. Because $\lim_{n\to\infty} |s_n| = \infty$, there exists $s_w \in M$ such that $|s_u s_w| = |s_v s_w| > \max\{|s_{n_5}|, |s_j| : j \in N_5\}$. But, $s_u s_w = s_a$ and $s_v s_w = s_b$ are distinct elements of $M$ and so, we must have $a \neq b \geq n_5$ and then, by what we have seen, $|s_a| \neq |s_b|$, a contradiction. Thus, $|s_u| \neq |s_v|$ whenever $u \neq v$.

Next, we assert that, in this case, $M$ is cyclic. The elements $s_u$ of $M$ are monotonic increasing in modulus (to $\infty$) for $u \notin N_5$, while $N_5$ is finite. In addition, $|s_u| \neq |s_v|$ if $u \neq v$. Thus, there exists a unique $w \in M$ of least modulus greater than 1. Suppose $M \neq < w >$. Then, there exists a unique $w_1 \in M\backslash < w >$ of least modulus greater than 1. Let $t$ be the largest natural number such that $1 < |w^t| < |w_1|$. Certainly $\dfrac{w_1}{w^t} \in M\backslash < w >$, whereas $1 < \left|\dfrac{w_1}{w^t}\right| < |w_1|$, contrary to choice. Thus, $M =< w >$ is cyclic with $|w| > 1$ as above.

We now show that there exists $i \in \{1, \ldots, k\}$ such that $M =< g_i >$. The $|s_u|$ are all distinct and are monotonic increasing from $n_5$ onwards. If $v \geq n_5$, then $s_{v+1} \in M$ must have the form $s_v w^j$ for some $j$, but $|s_v| < |s_v w| \leq |s_v w^j|$ and so $j = 1$. Recall, from [2], that $\boldsymbol{s}_{u,k}$ denotes the $k$-vector $(s_u, \ldots, s_{u+k-1})$. Fix $x \geq n_5$. Then, $\boldsymbol{s}_{x+1,k} = w\boldsymbol{s}_{x,k}$ and

$$\boldsymbol{s}_{x,k} = (w^t, \ldots, w^{t+k-1})$$

for some $t = t(x) \in \mathbb{Z}$. Then from [2], we have

$$\boldsymbol{s}_{x,k} C_f = \boldsymbol{s}_{x+1,k} = w\boldsymbol{s}_{x,k},$$

where $C_f$ is the companion matrix of $f(t)$, so that

$$(w^t, \ldots, w^{t+k-1})C_f = w(w^t, \ldots, w^{t+k-1}).$$

But, $w$ is a nonzero scalar, while $C_f$ is invertible and so,

$$w^{-1}(w^t, \ldots, w^{t+k-1}) = (w^t, \ldots, w^{t+k-1})C_f^{-1}.$$

We are assuming that $s_0 = 1$. Then, $\boldsymbol{s}_{0,k} = (1, s_1, \ldots, s_{k-1})$ and, with $x$ and $t$ as above,

$$\boldsymbol{s}_{0,k} C_f^x = \boldsymbol{s}_{x,k},$$

so that,

$$(w^{t-x}, \ldots) = \boldsymbol{s}_{x,k} C_f^{-x} = \boldsymbol{s}_{0,k} = (1, \ldots).$$

Thus, $t = x$ and $\boldsymbol{s}_{0,k} = (1, w, \ldots, w^{k-1})$, while $\boldsymbol{s}_{1,k} = (w, w^2, \ldots, w^k)$. But then,

$$w^k = s_k = a_{k-1}s_{k-1} + \cdots + a_0 s_0 = a_{k-1}w^{k-1} + \cdots + a_0 1$$

and so, $w$ is a root of $f(t)$, whence $w = g_i$ for some $i \in \{1, \ldots, k\}$. But then, $\alpha_i = 1$ and $\alpha_j = 0$ for all $j \neq i$, a contradiction.

Case (5). Finally, suppose $|g_{k_2}| < |g_{k_1}| < 1$. Then, the monic reciprocal, $f^*(t)$, falls under Case 4 and we again reach a contradiction.

It follows that $M$ is standard. The final assertion is now clear. □

## 3. Some Nonstandard Subgroups

In this section, we present some families of nonstandard subgroups in $\mathbb{C}^*$.

**Theorem 3.1.** *Let $p(t) \in \mathbb{C}_0[t]$, let $d \in \mathbb{N}$ with $d \geq 2$, and let $f(t) = p(t^d)$. Suppose, for some $m \in \mathbb{N}$, $g \in \mathbb{C}^*$ is a root of $f(t)$ of multiplicative order $m$. Write $M = < g > \leq \mathbb{C}^*$.*

*(a) Write $b = \gcd(m, d)$. Then, $< g^b > = < g^d >$ is a $p$-subgroup of order $m/b$. Let $\tau$ be the number of cyclically distinct $p$-sequences that represent $< g^b >$. Then, there exist at least*

$$\theta = \tau^b (b-1)! \left(\frac{m}{b}\right)^{b-1}$$

*cyclically distinct $f$-sequences that represent $M$.*

*(b) Suppose $d \mid m$. Furthermore, suppose that $m > 3$, if $d = 3$ and $m > 4$, if $d = 2$. Then, $M$ is a nonstandard $f$-subgroup of the first type.*

*Proof.* (a) Because $g$ is a root of $f(t)$, $g^d$ is a root of $p(t)$ and $< g^d > = < g^b > \leq \mathbb{C}^*$ is a $p$-subgroup of order $m/b$. By hypothesis, there are $\tau$ cyclically-distinct $p$-sequences of the form

$$S_1 = (\ldots, 1, c_1, c_2, \ldots, c_{\frac{m}{b}-1}, \ldots)$$

that represent $< g^b >$; in a similar way, for any $a \in M$, there are $\tau$ $p$-sequences

$$S_a = (\ldots, a, ac_1, ac_2, \ldots, ac_{\frac{m}{b}-1}, \ldots),$$

that represent (in the obvious way) the coset $a < g^b >$. Interleaving the $b$ $p$-sequences $S_1$, $S_{a_1}$, $\ldots$, $S_{a_{b-1}}$, where $\{1, a_1, \ldots, a_{b-1}\}$ is a complete set of coset representatives of $< g^b >$ in $M$, we may now represent $M$ by $f$-sequences of the form

$$S = (\ldots, 1, a_1, \ldots, a_{b-1}, c_1, a_1c_1, \ldots, a_{b-1}c_1, c_2, a_1c_2, \ldots, c_{\frac{m}{b}-1}, a_1c_{\frac{m}{b}-1}, \ldots, a_{b-1}c_{\frac{m}{b}-1}, \ldots)$$

having a minimal periodic segment of length $m$.

There are $\tau$ choices for $S_1$, $(m - \frac{m}{b})\tau$ choices for $a_1$ , and so $(m - \frac{m}{b})\tau = \frac{m}{b}(b-1)\tau$ for $S_{a_1}$, $(m - 2\frac{m}{b})\tau = \frac{m}{b}(b-2)\tau$ for $S_{a_2}$, and so on. Thus, the total number of choices is given by

$$\theta = \tau\left(\frac{m}{b}(b-1)\tau\right)\left(\frac{m}{b}(b-2)\tau\right)\ldots\left(\frac{m}{b}\tau\right) = \tau^b\left(\frac{m}{b}\right)^{b-1}(b-1)!$$

It follows that the group $M$ can be represented by at least $\theta$ cyclically-distinct sequences (and so will be nonstandard if $\theta > k_m$).

(b) Let $k_m \in \mathbb{N}$ denote the number of roots of $f(t)$ of multiplicative order $m$. By hypothesis, $d \mid m$ and so, $b = d$ and $m/d \geq 1$. Because $f$ has $k_m$ roots of order $m$ (including $g$), $p$ has $h_m \geq k_m/d$ roots of order $m/d$ (including $g^d$) in $\mathbb{C}$. Each of these roots of order $m/d$ will afford a $p$-sequence that represents $< g^d >$ (because $\mathbb{C}^*$ has a unique subgroup of each finite order) and so, $\tau \geq h_m \geq k_m/d$. Thus, in the notation of (a) and because $h_m \geq 1$, we have

$$\theta = h_m h_m^{d-1} \left(\frac{m}{d}\right)^{d-1} (d-1)! \geq \frac{k_m}{d} h_m^{d-1} \left(\frac{m}{d}\right)^{d-1} (d-1)! \geq k_m \frac{(d-1)!}{d}.$$

If $d \geq 4$, then $\theta > k_m$ and so $M$ is nonstandard as an $f$-subgroup.

Next, suppose $d = 3$. Then,

$$\theta \geq \frac{k_m}{3} \left(\frac{m}{3}\right)^2 2! = k_m \frac{2m^2}{27}.$$

By hypothesis, $m > 3$ in this case, so $\theta > k_m$ and $M$ is nonstandard.

Finally, suppose $d = 2$. Then, $\theta \geq k_m \frac{m}{4}$ and, because $m > 4$ in this case again, $M$ is nonstandard.

In all cases, $M$ has a cyclic representation and is nonstandard of the first type. □

**Example 3.2.** *Let $p(t) = t - i \in \mathbb{C}_0[t]$ and $f(t) = p(t^3)$. The roots of $f(t)$ are $g_1 = e^{\frac{\pi}{6}i}$ and $g_2 = e^{\frac{5\pi}{6}i}$, both of order 12, and $g_3 = e^{\frac{3\pi}{2}i}$ of order 4; thus, $k_{12} = 2$. Let $M = < g_1 > \leq \mathbb{C}^*$. By Theorem 3.1(b), $M$ is nonstandard and, by Theorem 3.1(a), there are at least $\theta = 32$ distinct $f$-sequences that represent $M$.*

**Theorem 3.3.** *Let $p(t) \in \mathbb{C}_0[t]$, let $d \in \mathbb{N}$ with $d \geq 2$, and let $f(t) = p(t^d)$. Suppose $h \in \mathbb{C}^*$ is a root of $p$. Let $g$ be a fixed dth root of $h$ and suppose $g$ is not a root of unity.*

*(a) There exist infinitely many cyclically-distinct $f$-sequences that represent $M =< g >$ and $M$ is a nonstandard $f$-subgroup of the first type.*

*(b) Let $\zeta \in \mathbb{C}^*$ be a primitive dth root of unity and write $N =< \zeta > \times < h >$. Then, $N$ is a nonstandard $f$-subgroup of the second type, represented by infinitely many $f$-sequences.*

*Proof.* (a) Write $S_0 = (g^i)_{i\in\mathbb{Z}}$; then, $S_0$ is a cyclic $f$-sequence that represents $M$. Fix $m \in \mathbb{Z}$, $m \geq 1$. Let $\mu_i = g^i$ if $d$ does not divide $i$, and $\mu_{qd} = g^{(q+m)d}$ for $q \in \mathbb{Z}$. Then, $S_m = (\mu_i)_{\in\mathbb{Z}}$ is an $f$-sequence that represents $M$ and is noncyclic, because $d \geq 2$, and $g$ is not a root of unity. Thus, $M$ is nonstandard of the first type. Distinct $m_1, m_2 \in \mathbb{Z}$ give cyclically distinct sequences $S_{m_1}$ and $S_{m_2}$, so $M$ is represented by an infinite number of $f$-sequences.

(b) The $f$-sequence

$$(s_i)_{i\in\mathbb{Z}} = (\ldots, 1, \zeta^1, \ldots, \zeta^{d-1}, h, h\zeta^1, \ldots, h\zeta^{d-1}, h^2, h^2\zeta^1, \ldots)$$

represents the group $N = < \zeta > \times < h > \leq \mathbb{C}^*$, which is an $f$-subgroup; it is clearly noncyclic and is nonstandard of the second type. The above sequence may be viewed as $d$ sequences interleaved; we may "slide" one of these interleaved sequences past the others and so, $N$ may be represented by infinitely many $f$-sequences. □

**Example 3.4.** *Let $p(t) = t^2 - t - 1$ and $f(t) = p(t^2)$ and write $\varphi$ for the golden ratio. Then,*

$$M = < \sqrt{\varphi} > = (\ldots, 1, \sqrt{\varphi}, \varphi, \varphi\sqrt{\varphi}, \varphi^2 \ldots)$$

*and*

$$N =< -1 > \times < \varphi > = (\ldots, -1, 1, -\varphi, \varphi, -\varphi^2, \varphi^2, \ldots)$$

*are infinite $f$-subgroups, nonstandard of the first and second type, respectively.*

We end with a result concerning a special class of polynomials.

**Theorem 3.5.** *Let $f(t) = t^k + t^{k-1} + t^{k-2} + \cdots + t + 1 \in \mathbb{C}_0[t]$, $k \in \mathbb{N}$. Let $\zeta$ be a (chosen, fixed) primitive $(k+1)$th root of unity in $\mathbb{C}$. Then, $M =< \zeta >$ is a nonstandard $f$-subgroup of the first type, if $k > 2$ and is standard, if $k = 2$.*

*Proof.* The roots of $f(t)$ are the elements of the set $\{\zeta, \zeta^2, \ldots, \zeta^k\}$. Any sequence $\mathcal{S} = (s_i)_{i \in \mathbb{Z}}$ in $\mathbb{C}$ such that

$$s_{n+k} + s_{n+k-1} + s_{n+k-2} + \cdots + s_n = 0$$

for all $n \in \mathbb{Z}$ is an $f$-sequence. Thus, whenever

$$\{\zeta_1, \zeta_2, \ldots, \zeta_k\} = \{\zeta, \zeta^2, \ldots, \zeta^k\},$$

the sequence

$$(\ldots, 1, \zeta_1, \zeta_2, \ldots, \zeta_{k-1}, \zeta_k, 1, \ldots)$$

is an $f$-sequence that represents $M$, and there are $k!$ ways in which this may occur. Therefore, if $k > 2$, $M$ is nonstandard because $k! > k$, and is of the first type because, at least one of these representations is cyclic. If $k = 2$, then $k! = k$, and in that case, $M$ is standard. □

## References


[1] O. J. Brison and J. E. Nogueira, *Linear recurring sequence subgroups in the complex field*, The Fibonacci Quarterly, **41.5** (2003), 397–404.

[2] O. J. Brison and J. E. Nogueira, *Matrices and linear recurrences in finite fields*, The Fibonacci Quarterly, **44.2** (2006), 103–108.

[3] L. E. Somer, *The Fibonacci group and a new proof that $F_{p-\frac{5}{p}} \equiv 0 \pmod{p}$*, The Fibonacci Quarterly, **10.3** (1972), 345–348 and 354.

[4] L. E. Somer, *Fibonacci-like groups and periods of Fibonacci-like sequences*, The Fibonacci Quarterly, **15.1** (1977), 35–41.


MSC2010: 11B39, 33C05


Departamento de Matemática, Faculdade de Ciências da Universidade de Lisboa, Bloco C6, Piso 2, Campo Grande, 1749-016 LISBOA, PORTUGAL

*E-mail address*: ojbrison@fc.ul.pt

Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Quinta da Torre, 2825-114 MONTE DA CAPARICA, PORTUGAL

*E-mail address*: jen@fct.unl.pt