JACOBSTHAL AND JACOBSTHAL-LUCAS WALKS

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Abstract. We construct digraph models for Jacobsthal and Jacobsthal-Lucas walks; extract as byproducts results for Pell, Pell-Lucas, Vieta, Vieta-Lucas, and Chebyshev polynomials; and explore some special classes of Jacobsthal and Jacobsthal-Lucas walks.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary complex variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [2, 7, 10].

In particular, *Pell polynomials* $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. The *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [6, 7, 10].

Let $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial* [3, 10]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$.

Suppose $a(x) = x$ and $b(x) = -1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = V_n(x)$, the nth *Vieta polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = v_n(x)$, the nth *Vieta-Lucas polynomial* [4, 8].

Let $a(x) = 2x$ and $b(x) = -1$. When $z_0(x) = 1$ and $z_1(x) = x$, $z_n(x) = T_n(x)$, the nth *Chebyshev polynomial of the first kind*; and when $z_0(x) = 1$ and $z_1(x) = 2x$, $z_n(x) = U_n(x)$, the nth *Chebyshev polynomial of the second kind* [6, 10].

1.1. Links Among the Subfamilies.

Fibonacci, Pell, and Jacobsthal polynomials, and Chebyshev polynomials of the second kind are closely linked; and so are Lucas, Pell-Lucas, and Jacobsthal-Lucas polynomials, and Chebyshev polynomials of the first kind [4, 7, 10]:

$$\begin{array}{rclcrcl} J_n(x) & = & x^{(n-1)/2} f_n(1/\sqrt{x}) & \quad & j_n(x) & = & x^{n/2} l_n(1/\sqrt{x}) \\ V_n(x) & = & i^{n-1} f_n(-ix) & & v_n(x) & = & i^n l_n(-ix) \\ V_n(x) & = & U_{n-1}(x/2) & & v_n(x) & = & 2T_n(x/2), \end{array}$$

where $i = \sqrt{-1}$.

In the interest of brevity and convenience, we *omit* the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$.

2. Jacobsthal Walks

A *digraph* (*di*rected *graph*) is a graph with n vertices $v_1, v_2, \ldots, v_n$, and directed edges connecting them. When there is a unique edge from v_i to v_j, it is denoted by v_i–v_j, i–j, or by the *word* ij for brevity.

A *walk* (or *directed path*) from v_i to v_j in a connected digraph is a sequence v_i–e_i–v_{i+1}–$\cdots$–v_{j-1}–e_{j-1}–v_j of vertices v_k and directed edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is *closed* if its endpoints are the same; otherwise, it is *open*. The *length* ℓ of a walk is the number of edges in the walk; that is, it takes ℓ steps to reach from one endpoint to the other.

Consider a walk originating at the origin and consisting of n unit steps in the *easterly* direction. Such a unit step is an E-step. A D-step (D for *double*) is made up of two E-steps. Now, assign a *weight* to each step, 1 to an E-step and x to a D-step. The weight of a walk is the product of the weights of all steps in it. The weight of the walk of length 0 is defined as 1. Such a walk is a *Jacobsthal walk* of *length n*.

Figure 1 shows Jacobsthal walks of length 5, where a thick dot indicates the origin, and directions are *omitted* for convenience. The sum of the weights of all those walks is $3x^2+4x+1 = J_6(x)$.

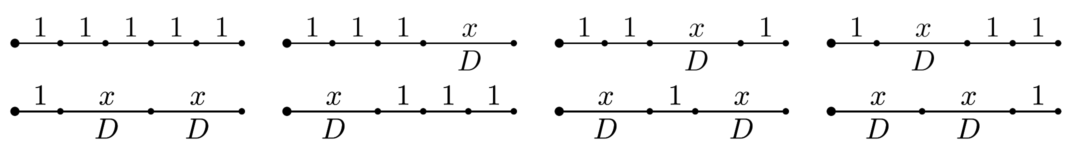

Figure 1. Jacobsthal Walks of Length 5

Let $S_n(x)$ denote the sum of the weights of Jacobsthal walks of length n. Clearly, $S_0(x) = 1 = J_1(x)$ and $S_1(x) = 1 = J_2(x)$. Now, consider an arbitrary walk of length $n \geq 2$. Since it can end in an E-step or a D-step, it follows that $S_n(x) = S_{n-1}(x)+xS_{n-2}(x)$. This recurrence, coupled with the initial conditions, implies that $S_n(x) = J_{n+1}(x)$, where $n \geq 0$. Thus, we have the following result.

Theorem 2.1. *The sum of the weights of Jacobsthal walks of length n is $J_{n+1}(x)$, where $n \geq 0$.*

This implies the next result.

Corollary 2.2. *There are F_{n+1} Jacobsthal walks of length n and the sum of the weights of Jacobsthal walks of length n is J_{n+1} when the weight of a D-step is 2, where $n \geq 0$.*

Let E denote an E-step and D a D-step. Then a Jacobsthal walk of length n can be denoted by a word of length at most n; each such word contains Es or Ds; or x's or 1s.

For example, the Jacobsthal walks in Figure 1 can be represented by the following words:

$$\begin{array}{llll} 11111 & 111x & 11x1 & 1x11 \\ 1xx & x111 & x1x & xx1. \end{array}$$

When $x = 2$, they yield the F_6 *compositions* of the positive integer 5 using the summands 1 and 2 [9]:

$$\begin{array}{llll} 1+1+1+1+1 & 1+1+1+2 & 1+1+2+1 & 1+2+1+1 \\ 1+2+2 & 2+1+1+1 & 2+1+2 & 2+2+1. \end{array}$$

Using Jacobsthal walks, we can establish some delightful properties of Jacobsthal polynomials. The next three theorems [10] show such results. Their proofs are straightforward.

Theorem 2.3. $J_{n+1}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^k$, *where* $n \geq 0$.

This follows by counting k, the number of D-steps in walks of length n.

$n \backslash k$	0	1	2	3	4
0	1				
1	1				
2	1	x			
3	1	$2x$			
4	1	$3x$	x^2		
5	1	$4x$	$3x^2$		
6	1	$5x$	$6x^2$	x^3	
7	1	$6x$	$10x^2$	$4x^3$	
8	1	$7x$	$15x^2$	$10x^3$	x^4
9	1	$8x$	$21x^2$	$20x^3$	$5x^4$

TABLE 1. Array A

We can employ Theorem 2.3 to construct the triangular array $A = (a_{n,k})$ in Table 1, where $a_{n,k} = a_{n,k}(x)$ and $0 \leq k \leq \lfloor n/2 \rfloor$. Clearly, $a_{n,k} = a_{n-1,k} + xa_{n-1,k-1}$ (see the arrows arrows in the table), where $a_{0,0} = 1$ and $a_{1,1} = 0$.

2.1. **A Hidden Treasure.** Array A contains a hidden treasure. When $x = 1$, the resulting array occurs in the study of the paraffins C_nH_{2n+2}. To see this, delete the hydrogen atoms from their structural formulas (geometric representations); this yields a *path graph* P_n with n vertices. The *topological index* of P_n is the total number of ways of partitioning it into k disjoint subgraphs with k edges, where $k \geq 0$ [5, 9]; $a_{n,k}(1)$ is the number of Jacobsthal walks of length n with exactly k D-steps. The *topological index* of the paraffin is $\sum_{k \geq 0} a_{n,k}(1) = F_{n+1}$.

For example, Figure 2 shows the structural formula of the hydrocarbon molecule C_4H_{10}, namely, butane; it contains 4 carbon atoms and 10 hydrogen atoms. Its topological index is 5.

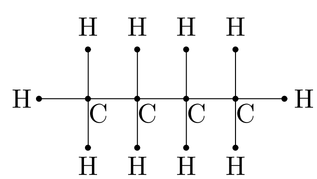

FIGURE 2. Butane Molecule C_4H_{10}

2.2. **Breakability.** To establish the *addition formula* for Jacobsthal polynomials in the next theorem, we introduce the concept of *breakability* [1, 9, 10]. A Jacobsthal walk of length n is *unbreakable* at step k if a D-step occupies unit steps k and $k+1$; otherwise, it is *breakable* at k. For example, the walk in Figure 3 is unbreakable at steps 2 and 4, and breakable at unit steps 0, 1, 3, 5, 6, and 7. (The M in the figure is explained later.)

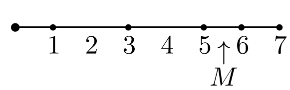

FIGURE 3. Walk Unbreakable at 2 and 4

Theorem 2.4. *Let $m, n \geq 1$. Then,* $J_{m+n}(x) = J_{m+1}(x)J_n(x) + xJ_m(x)J_{n-1}(x)$ [10].

The proof of this theorem follows by considering breakability at unit step m.

It follows from Theorem 2.4 that [10]

$$\begin{aligned} J_{2n}(x) &= J_n(x)\left[J_{n+1}(x) + xJ_{n-1}(x)\right] \\ &= J_n(x)j_n(x); \\ J_{2n+1}(x) &= J_{n+1}^2(x) + xJ_n^2(x). \end{aligned}$$

Next, we introduce the concept of the median step.

2.3. **Median Step.** Suppose the length of a Jacobsthal walk W is odd. Then W must have an odd number of E-steps. So W contains a special E-step M with the same number of E-steps on either side of it; M is the *median step* of the walk. For example, the E-step at 6 in Figure 3 is the median step of that walk.

We can employ the concept of the median step to derive a charming formula [10] for $J_{2n+2}(x)$, as the next theorem demonstrates.

Theorem 2.5. *Let $n \geq 0$. Then* $J_{2n+2}(x) = \sum_{\substack{i,j\geq 0\\ i+j\leq n}} \binom{n-i}{j}\binom{n-j}{i} x^{i+j}$.

Proof. Consider an arbitrary Jacobsthal walk W of length $2n+1$. By Theorem 2.1, the sum S of the weights of all such walks is $J_{2n+2}(x)$.

We will now compute S in a different way. Since the length of the walk is odd, W contains a median E-step. Suppose there are i D-steps to the left of M and j D-steps to its right. Then W contains a total of $(2n+1) - (2i+2j) = 2n - 2i - 2j + 1$ E-steps; so there are $n - i - j$ E-steps on either side of M. Consequently, there are $n - j$ steps to the left of M and $n - i$ steps to its right:

$$\underbrace{\overbrace{\text{... E ... E ...}}^{n-i-j\ \text{E-steps}}}_{n-j\ \text{steps}}\ \underset{\substack{\uparrow\\ M}}{\text{E}}\ \underbrace{\overbrace{\text{... E ... E ...}}^{n-i-j\ \text{E-steps}}}_{n-i\ \text{steps}}.$$

The $n-i-j$ E-steps to the left of M can be placed among the $n-j$ steps in $\binom{n-j}{n-i-j} = \binom{n-j}{i}$ different ways; the sum of the weights of such subwalks is $\binom{n-j}{i}x^i$. Likewise, the sum of the weights of subwalks to the right of M is $\binom{n-i}{j}x^j$. Thus, the cumulative sum S of the weights of all walks W also equals

$$\sum_{\substack{i,j\geq 0\\ i+j\leq n}} \binom{n-j}{i}x^i \cdot 1 \cdot \binom{n-i}{j}x^j = \sum_{\substack{i,j\geq 0\\ i+j\leq n}} \binom{n-i}{j}\binom{n-j}{i}x^{i+j}.$$

Equating the two values of S yields the desired result. □

In particular, we have

$$\begin{aligned} F_{2n+2} &= \sum_{\substack{i,j\geq 0\\ i+j\leq n}} \binom{n-i}{j}\binom{n-j}{i}; \\ J_{2n+2} &= \sum_{\substack{i,j\geq 0\\ i+j\leq n}} \binom{n-i}{j}\binom{n-j}{i}2^{i+j}. \end{aligned}$$

It follows by the Jacobsthal-Fibonacci relationship $J_n(x) = x^{(n-1)/2}f_n(1/\sqrt{x})$ that $f_n = x^{n-1}J_n(1/x^2)$. Consequently, Theorem 2.5 has a Fibonacci counterpart, as the next corollary shows.

Corollary 2.6. $f_{2n+2} = \sum_{\substack{i,j\geq 0\\ i+j\leq n}} \binom{n-i}{j}\binom{n-j}{i} x^{2n-2i-2j+1}$.

This implies

$$p_{2n+2} = \sum_{\substack{i,j\geq 0\\ i+j\leq n}} \binom{n-i}{j}\binom{n-j}{i} (2x)^{2n-2i-2j+1};$$

$$P_{2n+2} = \sum_{\substack{i,j\geq 0\\ i+j\leq n}} \binom{n-i}{j}\binom{n-j}{i} 2^{2n-2i-2j+1}.$$

2.4. **Some Special Jacobsthal Walks.** It follows from Theorem 2.1 that the sum of the weights of Jacobsthal walks of length n that begin with:

- E is $J_n(x)$, where $n \geq 1$.
- E and end in E is $J_{n-1}(x)$, where $n \geq 1$.
- D is $xJ_{n-1}(x)$, where $n \geq 2$.
- D and end in D is $x^2J_{n-3}(x)$, where $n \geq 4$.
- D and end in E is $xJ_{n-2}(x)$, where $n \geq 3$.

Next, we construct a digraph model for Jacobsthal-Lucas polynomials $j_n(x)$.

3. Jacobsthal-Lucas Walks

Here also, a walk contains E- or D-steps. The weight of the walk of length 0 is 1. The weight of an E-step is 1 unless it appears at the beginning of the walk, in which case its weight is $w = 2x+1$. Such a walk is a *Jacobsthal-Lucas walk*.

Figure 4 shows the Jacobsthal-Lucas walks of length 5. The sum of the weights of all such walks is $2x^3 + 9x^2 + 6x + 1 = j_6(x)$.

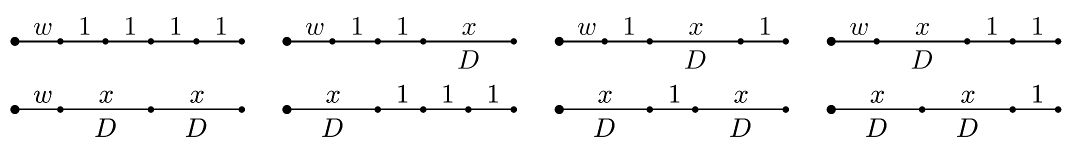

Figure 4. Jacobsthal-Lucas Walks of Length 5

Let $S_n(x)$ denote the sum of the weights of Jacobsthal-Lucas walks W of length n. Clearly, $S_0(x) = 1 = j_1(x)$ and $S_1(x) = 2x + 1 = j_2(x)$. Now, consider an arbitrary walk of length $n \geq 2$. Here also, $S_n(x)$ satisfies the Jacobsthal recurrence; so $S_n(x) = j_{n+1}(x)$. Thus, we have the following result.

Theorem 3.1. *The sum of the weights of Jacobsthal-Lucas walks of length n is $j_{n+1}(x)$, where $n \geq 0$.*

This implies the next result.

Corollary 3.2. *There are L_{n+1} Jacobsthal-Lucas walks of length n; and the sum of the weights of Jacobsthal walks of length n is j_{n+1} when the weight of a D-step is 2, where $n \geq 0$.*

3.1. **Some Special Jacobsthal-Lucas Walks.** It follows from Theorem 2.1 that the sum of the weights of Jacobsthal-Lucas walks of length n that begin with:

- E is $(2x+1)J_n(x)$, where $n \geq 1$.
- D is $xJ_{n-1}(x)$, where $n \geq 2$.
- D and end in D is $x^2J_{n-3}(x)$, where $n \geq 4$.

It then follows that we can express $j_{n+1}(x)$ in terms of $J_n(x)$ and $J_{n-1}(x)$, as the next theorem [10] shows.

Theorem 3.3. $j_{n+1}(x) = (2x+1)J_n(x) + xJ_{n-1}(x)$, *where* $n \geq 0$.

It follows from the relationship $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ that $l_n = x^{n/2} j_n(1/x^2)$. Consequently, this theorem implies that $l_{n+1} = (x^2+2)f_n + xf_n$ and $j_{n+1} = 5J_n + 2J_{n-1}$.

Using the Jacobsthal recurrence, we can rewrite the formula in Theorem 3.3:

$$\begin{aligned} j_{n+1}(x) &= J_{n+1}(x) + 2xJ_n(x) \\ &= J_{n+2}(x) + xJ_n(x). \end{aligned}$$

Using this formula, we can construct an array similar to array A. Table 2 shows the resulting array $B = (b_{n,k})$, where $b_{n,k} = b_{n,k}(x), 0 \leq k \leq \lfloor n/2 \rfloor$, and $n \geq 1$. Clearly, $b_{n,k} = b_{n-1,k} + xb_{n-1,k-1}$ (see arrows in the table), where $b_{1,0} = 1$ and $b_{2,1} = 2x$.

$n \backslash k$	0	1	2	3	4	5
1	1					
2	1	$2x$				
3	1	$3x$				
4	1	$4x$	$2x^2$			
5	1	$5x$	$5x^2$			
6	1	$6x$	$9x^2$	$2x^3$		
7	1	$7x$	$14x^2$	$7x^3$		
8	1	$8x$	$20x^2$	$16x^3$	$2x^4$	
9	1	$9x$	$27x^2$	$30x^3$	$9x^4$	
10	1	$10x$	$35x^2$	$50x^3$	$25x^4$	$2x^5$

TABLE 2. Array B

The row sum $\sum_{k \geq 0} b_{n,k}(1)$ gives the topological index L_n of the cycloparaffin C_nH_{2n}, where $n \geq 1$ [5, 9]. For example, Figure 5 shows the structural formula of the hydrocarbon molecule cyclobutane C_4H_8; its topological index is 7.

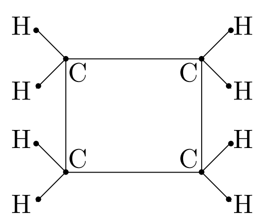

FIGURE 5. Cyclobutane Molecule C_4H_8

The next theorem [10], an alternate version of Theorem 3.3, gives the Jacobsthal-Lucas counterpart of Theorem 2.3.

Theorem 3.4.

$$j_{n+1}(x) = (2x+1) \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{k} x^k + \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} \binom{n-k-2}{k} x^{k+1},$$

where $n \geq 0$.

Proof. Let W be an arbitrary Jacobsthal-Lucas walk of length n. By Theorem 3.1, the sum S of the weights of all such walks is $j_{n+1}(x)$.

To compute S in a different way, assume W contains k D-steps. Suppose W begins with an E-step: E $\underbrace{\text{subwalk}}_{\text{length } n-1}$. The subwalk involves $n-k-1$ steps, of which k are D-steps. The k D-steps can be placed among the $n-k-1$ steps in $\binom{n-k-1}{k}$ ways; so the sum S_1 of the weights of such walks is $S_1 = (2x+1)\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{k} x^k$.

On the other hand, suppose W begins with a D-step: D $\underbrace{\text{subwalk}}_{\text{length } n-2}$. The subwalk contains $n-k-1$ steps; $k-1$ of them are D-steps and can be placed among them in $\binom{n-k-1}{k-1}$ ways. The sum S_2 of the weights of such walks equals

$$\begin{aligned} S_2 &= \sum_{k\geq 0} \binom{n-k-1}{k-1} x^k \\ &= \sum_{k\geq 0}^{\lfloor (n-2)/2 \rfloor} \binom{n-k-2}{k} x^{k+1}. \end{aligned}$$

Thus, $S = S_1 + S_2$. This yields the given result. □

It follows from this theorem that

$$\begin{aligned} L_{n+1} &= 3\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{k} + \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} \binom{n-k-2}{k} \\ &= 3F_n + F_{n-1}; \\ j_{n+1} &= 5\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{k} 2^k + \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} \binom{n-k-2}{k} 2^{k+1} \\ &= 5J_n + 2J_{n-1}, \end{aligned}$$

as we saw earlier.

By invoking breakability and Theorem 3.1, we can establish the *addition formula* for Jacobsthal-Lucas polynomials:

$$j_{m+n}(x) = j_{m+1}(x)J_n(x) + xj_m(x)J_{n-1}(x).$$

It then follows that

$$\begin{aligned} j_{2n}(x) &= j_{n+1}(x)J_n(x) + xj_n(x)J_{n-1}(x); \\ j_{2n+1}(x) &= j_{n+1}(x)J_{n+1}(x) + xj_n(x)J_n(x); \\ j_{2n} &= j_{n+1}J_n + 2j_nJ_{n-1}; \\ j_{2n+1} &= j_{n+1}J_{n+1} + 2j_nJ_n \\ &= J_{2n+2} + 2J_{2n}. \end{aligned}$$

Using the concept of the median E-step in a Jacobsthal-Lucas walk of odd length, we can establish the following counterpart of Theorem 2.5.

Theorem 3.5.

$$j_{2n+2}(x) = \sum_{i,j\geq 0} \left[\binom{n-j-1}{i}(2x+1) + \binom{n-j-1}{i-1} \right] \binom{n-i}{j} x^{i+j},$$

where $n \geq 0$.

The next result follows from this theorem by virtue of the relationship $l_n = x^n j_n(1/x^2)$.

Corollary 3.6.

$$l_{2n+2} = \sum_{i,j\geq 0} \left[\binom{n-j-1}{i}(x^2+2) + \binom{n-j-1}{i-1} x^2 \right] \binom{n-i}{j} x^{2n-2i-2j},$$

where $n \geq 0$.

It also follows from this theorem that

$$\begin{aligned} j_{2n+2} &= \sum_{i,j\geq 0} \left[5\binom{n-j-1}{i} + \binom{n-j-1}{i-1}\right]\binom{n-i}{j}2^{i+j} \\ &= \sum_{i,j\geq 0} \left[\binom{n-j}{i} + 4\binom{n-j-1}{i}\right]\binom{n-i}{j}2^{i+j}; \\ L_{2n+2} &= \sum_{i,j\geq 0} \left[3\binom{n-j-1}{i} + \binom{n-j-1}{i-1}\right]\binom{n-i}{j} \\ &= \sum_{i,j\geq 0} \left[\binom{n-j}{i} + 2\binom{n-j-1}{i}\right]\binom{n-i}{j}; \\ q_{2n+2} &= 2\sum_{i,j\geq 0} \left[\binom{n-j-1}{i}(2x^2+1) + 2\binom{n-j-1}{i-1}x^2\right]\binom{n-i}{j}(2x)^{2n-2i-2j}; \\ Q_{2n+2} &= \sum_{i,j\geq 0} \left[2\binom{n-j}{i} + \binom{n-j-1}{i}\right]\binom{n-i}{j}2^{2n-2i-2j}. \end{aligned}$$

4. Vieta and Chebyshev Consequences

Corollaries 2.6 and 3.6 have Vieta and Chebyshev implications. Since $V_n(x) = i^{n-1}f_n(-ix)$ and $v_n(x) = i^n l_n(-ix)$, where $i = \sqrt{-1}$, it follows that

$$\begin{aligned} V_{2n+2}(x) &= \sum_{j,k\geq 0} \binom{n-j}{k}\binom{n-k}{j}(-1)^{k+j}x^{2n-2j-2k+1}; \\ v_{2n+2}(x) &= (-1)^n \sum_{j,k\geq 0} \left[\binom{n-j-1}{k}(x^2-2) + \binom{n-j-1}{k-1}x^2\right](-ix)^{2n-2j-2k}. \end{aligned}$$

Using the relationships $V_n(x) = U_{n-1}(x/2)$ and $v_n(x) = 2T_n(x/2)$, we have

$$\begin{aligned} T_{2n+2}(x) &= (-1)^n \sum_{j,k\geq 0} \left[\binom{n-j-1}{k}(2x^2-1) + 2\binom{n-j-1}{k-1}x^2\right](-2ix)^{2n-2j-2k} \\ U_{2n+1}(x) &= \sum_{j,k\geq 0} \binom{n-j}{k}\binom{n-k}{j}(-1)^{k+j}(2x)^{2n-2j-2k+1}. \end{aligned}$$

Next, we explore a special class of Jacobsthal and Jacobsthal-Lucas walks.

5. Symmetric Jacobsthal Walks

A Jacobsthal walk of length n is *symmetric* if the corresponding word is palindromic. For example, the walk DED in Figure 1 is symmetric, whereas the walk EEDDE is not.

5.1. **Symmetric Jacobsthal Walks of Odd Length.** Consider an arbitrary Jacobsthal walk W of length $2n+1$. Then, W contains an odd number of E-steps and hence, a median E-step M. Let $S_n(x)$ denote the sum of the weights of such Jacobsthal walks. Clearly, $S_0(x) = 1 = J_1(x^2)$ and $S_1(x) = 1 = J_2(x^2)$.

Assume $S_{n-1}(x) = J_n(x^2)$, where $n \geq 2$. Let W be an arbitrary Jacobsthal walk of length $2n+1$. Suppose W begins with an E: E$\underbrace{\text{subwalk A}}_{\text{length } n-1}$ E $\underbrace{\text{subwalk B}}_{\text{length } n-1}$E. Noticing that subwalk B is the reflection of subwalk A, the sum of weights of such walks is $1 \cdot J_n(x^2) \cdot 1 = J_n(x^2)$. On the other hand, suppose W begins a D: D$\underbrace{\text{subwalk X}}_{\text{length } n-2}$ E $\underbrace{\text{subwalk Y}}_{\text{length } n-2}$D. Subwalk Y is the reflection of subwalk X, so the sum of the weights of such walks is $x \cdot J_{n-1}(x^2) \cdot x = x^2 J_{n-1}(x^2)$. Combining the two cases, we have $S_n(x) = J_n(x^2) + x^2 J_{n-1}(x^2) = J_{n+1}(x^2)$.

Thus, by induction, we have the following result.

Theorem 5.1. *The sum of the weights of all symmetric Jacobsthal walks of length $2n+1$ is $J_{n+1}(x^2)$, where $n \geq 0$.*

This yields the next result.

Corollary 5.2. *There are F_n symmetric Jacobsthal walks of length $2n+1$ that begin with E, and F_{n-1} such walks that begin with D. There are a total of F_{n+1} such walks.*

It also follows by the theorem that there are F_{n+1} palindromic compositions of the positive integer $2n+1$ using the summands 1 and 2 [9].

Next, we investigate symmetric Jacobsthal walks of even length.

5.2. **Symmetric Jacobsthal Walks of Even Length.** Let W be an arbitrary symmetric Jacobsthal walk of length $2n$. The number of Es in such a walk is even. So the middle step must be EE, D, or DD.

Let $S_n(x)$ denote the sum of the weights of such walks. Clearly, $S_0(x) = 1$ and $S_1(x) = x + 1$.

Let $n \geq 1$. We will now construct an algorithm to produce symmetric Jacobsthal walks of length $2n+2$ from those of lengths $2n$ and $2n-2$.

Step 1. Place an E at each end of the walks of length $2n$. This produces symmetric Jacobsthal walks of length $2n+2$, and the sum of their weights is $S_n(x)$.

Step 2. Place a D at each end of the walks of length $2n-2$. This step also creates symmetric Jacobsthal walks of length $2n+2$, and the sum of the weights such walks is $x^2 S_{n-1}(x)$.

Thus, the cumulative sum of the weights of all symmetric Jacobsthal walks of length $2n+2$ obtained by these two steps is $S_n(x) + x^2 S_{n-1}(x)$. Since the algorithm is reversible, it follows that $S_{n+1}(x) = S_n(x) + x^2 S_{n-1}(x)$, where $n \geq 1$, $S_0(x) = 1$, and $S_1(x) = x + 1$.

Consequently, there are F_{n+2} symmetric Jacobsthal walks of length $2n$, and hence, F_{n+2} palindromic compositions of the positive integer $2n$.

6. Symmetric Jacobsthal-Lucas Walks

Recall that the weight of an E-step is 1 except when the walk begins with it, in which case the weight is $w = 2x + 1$. Consequently, symmetric Jacobsthal-Lucas walks must begin with a D-step.

6.1. **Symmetric Jacobsthal-Lucas Walks of Odd Length.** Let W be an arbitrary symmetric Jacobsthal-Lucas walk of length $2n+1$. Since the number of Es in it must be odd, W must contain a median E: D $\underbrace{\text{subwalk A}}_{\text{length } n-2}$ E $\underbrace{\text{subwalk B}}_{\text{length } n-2}$ D. Since subwalk B is the reflection of subwalk A, it follows by Theorem 2.1 that the sum of the weights of such walks is $x^2 J_{n-1}(x^2)$. Thus, we have the following theorem.

Theorem 6.1. *The sum of the weights of all symmetric Jacobsthal-Lucas walks of length $2n+1$ is $x^2 J_{n-1}(x^2)$, where $n \geq 2$.*

This yields the next result.

Corollary 6.2. *There are F_{n-1} symmetric Jacobsthal-Lucas walks of length $2n+1$, where $n \geq 2$.*

6.2. **Symmetric Jacobsthal-Lucas Walks of Even Length.** Suppose the length of W is $2n$. Since its length is even, the number of Es in it must be even. To compute the sum of the weights of such walks, we focus on the parity of the number of Ds in W.

Case 1. Suppose the number of Ds is odd. Then, W has a unique median D: D $\underbrace{\text{subwalk}}_{\text{length } n-3}$ D $\underbrace{\text{subwalk}}_{\text{length } n-3}$ D.

By Theorem 2.1, the sum of the weights of such walks is $x^3 J_{n-2}(x^2)$.

Case 2. Suppose the number of Ds is even. Then, the middle can be EE or DD. If the middle is EE, then W must be of the form D $\underbrace{\text{subwalk}}_{\text{length } n-3}$ EE $\underbrace{\text{subwalk}}_{\text{length } n-3}$ D. Such walks contribute $x^2 J_{n-2}(x^2)$ toward the cumulative sum. On the other hand, if the middle is DD, then W has the form D $\underbrace{\text{subwalk}}_{\text{length } n-4}$ DD $\underbrace{\text{subwalk}}_{\text{length } n-4}$ D; the corresponding sum is $x^4 J_{n-3}(x^2)$.

Combining the two cases, the cumulative sum of the weights of all walks W is given by

$$\begin{aligned} x^3 J_{n-2}(x^2) + x^2 J_{n-2}(x^2) + x^4 J_{n-3}(x^2) &= x^2 \left[(x+1) J_{n-2}(x^2) + x^2 J_{n-3}(x^2)\right] \\ &= x^2 \left[x J_{n-2}(x^2) + J_{n-1}(x^2)\right]. \end{aligned}$$

Thus, we have the next theorem.

Theorem 6.3. *The sum of the weights of all symmetric Jacobsthal-Lucas walks of length* $2n$ *is* $x^2 \left[x J_{n-2}(x^2) + J_{n-1}(x^2)\right]$*, where* $n \geq 3$*.*

This yields the next result.

Corollary 6.4. *There are* F_n *symmetric Jacobsthal-Lucas walks of length* $2n$*, where* $n \geq 1$*.*

7. Ackowledgment

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