A RECURRENCE FOR GIBONACCI CUBES WITH GRAPH-THEORETIC CONFIRMATIONS

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ABSTRACT. We develop a fourth-order recurrence for gibonacci cubes, extend it to Pell and Jacobsthal families, and then confirm the recurrences using graph-theoretic tools.

1. INTRODUCTION

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary complex variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary complex polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 5, 7]. *Pell polynomials* $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. In particular, the *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = l_n(2)$, respectively [5].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial* [2, 7, 9]. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$ and $j_n(1) = L_n$.

Let $a(x) = x$ and $b(x) = -1$. When $g_0(x) = 0$ and $g_1(x) = 1$, $g_n(x) = V_n(x)$, the nth *Vieta polynomial*; and when $g_0(x) = 2$ and $g_1(x) = x$, $g_n(x) = v_n(x)$, the nth *Vieta-Lucas polynomial* [3, 7, 9].

Finally, let $a(x) = 2x$ and $b(x) = -1$. When $g_0(x) = 1$ and $g_1(x) = x$, $g_n(x) = T_n(x)$, the nth *Chebyshev polynomial of the first kind*; and when $g_0(x) = 1$ and $g_1(x) = 2x$, $g_n(x) = U_n(x)$, the nth *Chebyshev polynomial of the second kind* [3, 7, 9].

1.1. **Gibonacci Links.** The Jacobsthal, Vieta, and Chebyshev subfamilies are closely linked by the relationships in Table 1, where $i = \sqrt{-1}$ [3, 7, 9].

In the interest of clarity, concision, and convenience, we omit the argument in the functional notation, when there is no ambiguity; so z_n will mean $z_n(x)$. We let $g_n = f_n$ or l_n, $b_n = p_n$ or q_n, and $c_n = J_n(x)$ or $j_n(x)$. Correspondingly, let $G_n = F_n$ or L_n, $B_n = P_n$ or Q_n, and $C_n = J_n$ or j_n. We also omit a lot of basic algebra.

Next, we develop a fourth-order recurrence for gibonacci cubes g_n^3.

TABLE 1. Links Among the Subfamilies

$J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$	$j_n(x) = x^{n/2} l_n(1/\sqrt{x})$
$V_n(x) = i^{n-1} f_n(-ix)$	$v_n(x) = i^n l_n(-ix)$
$V_n(2x) = U_{n-1}(x)$	$v_n(2x) = 2T_n(x)$.

2. A Recurrence for Gibonacci Cubes

Using the gibonacci recurrence $g_{n+2} = xg_{n+1} + g_n$, we have

$$\begin{aligned}
g_{n+4}^3 &= (xg_{n+3} + g_{n+2})^3 \\
&= x^3 g_{n+3}^3 + 3x^2 g_{n+3}^2 g_{n+2} + 3xg_{n+3}g_{n+2}^2 + g_{n+2}^3 \\
&= x^3 g_{n+3}^3 + 2xg_{n+3}^2(g_{n+3} - g_{n+1}) + x^2 g_{n+3}^2 g_{n+2} + 3x(xg_{n+2} + g_{n+1})g_{n+2}^2 + g_{n+2}^3 \\
&= (x^3 + 2x)g_{n+3}^3 + g_{n+2}^3 - 2(xg_{n+2} + g_{n+1})^2(g_{n+2} - g_n) \\
&\quad + x^2 g_{n+2}(xg_{n+2} + g_{n+1})^2 + 3x(xg_{n+2} + g_{n+1})g_{n+2}^2 \\
&= x(x^2 + 2)g_{n+3}^3 + (x^4 + 3x^2 + 1)g_{n+2}^3 + 3xg_{n+2}^2 g_{n+1} - 3x^2 g_{n+2}g_{n+1}^2 - 2xg_{n+1}^3 \\
&= x(x^2 + 2)g_{n+3}^3 + (x^4 + 3x^2 + 1)g_{n+2}^3 + g_{n+2}^2(g_{n+2} - g_n) + 2xg_{n+2}^2 g_{n+1} \\
&\quad - 3x^2 g_{n+2}g_{n+1}^2 - 2xg_{n+1}^3 \\
&= x(x^2 + 2)g_{n+3}^3 + (x^4 + 3x^2 + 2)g_{n+2}^3 + E,
\end{aligned}$$

where

$$\begin{aligned}
E &= -g_{n+2}^2 g_n + 2xg_{n+2}^2 g_{n+1} - 3x^2 g_{n+2}g_{n+1}^2 - 2xg_{n+1}^3 \\
&= -(xg_{n+1} + g_n)^2 g_n + 2xg_{n+1}(xg_{n+1} + g_n)^2 - 3x^2 g_{n+1}^2(xg_{n+1} + g_n) - 2xg_{n+1}^3 \\
&= -x(x^2 + 2)g_{n+1}^3 - g_n^3.
\end{aligned}$$

Thus, we have the fourth-order recurrence

$$g_{n+4}^3 = x(x^2 + 2)g_{n+3}^3 + (x^2 + 1)(x^2 + 2)g_{n+2}^3 - x(x^2 + 2)g_{n+1}^3 - g_n^3. \tag{2.1}$$

In particular, we have

$$\begin{aligned}
G_{n+4}^3 &= 3G_{n+3}^3 + 6G_{n+2}^3 - 3G_{n+1}^3 - G_n^3; & (2.2)\\
b_{n+4}^3 &= 4x(2x^2 + 1)b_{n+3}^3 + 2(2x^2 + 1)(4x^2 + 1)b_{n+2}^3 - 4x(2x^2 + 1)b_{n+1}^3 - b_n^3; & (2.3)\\
B_{n+4}^3 &= 12B_{n+3}^3 + 30B_{n+2}^3 - 12B_{n+1}^3 - B_n^3.
\end{aligned}$$

Zeitlin and Parker discovered identity (2.2) with $G_n = F_n$ [6].

3. Graph-theoretic Models

Next, we confirm the gibonacci identity (2.1) with graph-theoretic tools. To this end, we introduce a *digraph* D_1 with two vertices v_1 and v_2, where a *weight* is assigned to each edge; see Figure 1 [8].

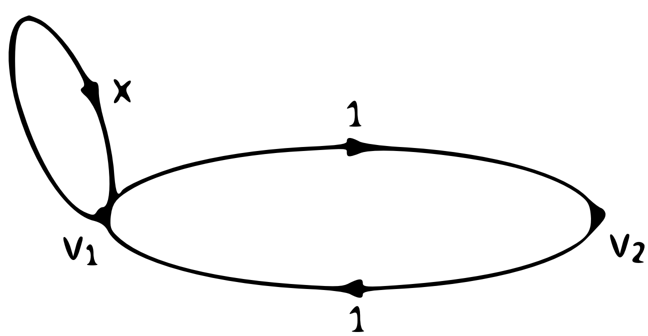

Figure 1. Weighted Digraph D_1

Its *weighted adjacency matrix* is

$$Q = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix},$$

where $Q = Q(x) = (q_{ij})_{2\times 2}$ [8]. It then follows by induction that

$$Q^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \geq 1$.

A *walk* from vertex v_i to vertex v_j is a sequence v_i-e_i-v_{i+1}-$\cdots$-v_{j-1}-e_{j-1}-v_j of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The walk is *closed* if $v_i = v_j$; otherwise, it is *open*. The *length* of a walk is the number of edges in the walk. The *weight* of a walk is the product of the weights of the edges along the walk.

We can employ the matrix Q^n to compute the weight of a walk of length n from any vertex v_i to any vertex v_j, as the following theorem shows [4, 8].

Theorem 3.1. *Let M be the weighted adjacency matrix of a weighted, connected digraph with vertices $v_1, v_2, \ldots, v_k$. Then, the ijth entry of the matrix M^n gives the sum of the weights of all walks of length n from v_i to v_j, where $n \geq 1$.*

The next result follows from this theorem.

Corollary 3.2. *The ijth entry of Q^n gives the sum of the weights of all walks of length n from v_i to v_j in the weighted digraph D_1, where $1 \leq i, j \leq n$.*

It follows by this corollary that the sum of the weights of all closed walks of length n originating at v_1 in the digraph is f_{n+1}, and that of walks of length n originating at v_2 is f_{n-1}. So, the sum of the weights of all closed walks of length n is $f_{n+1} + f_{n-1} = l_n$. These results play a central role in the confirmation proofs.

Part I. First, we will establish the equivalent identity with $g_n = f_n$:

$$f_{n+4}^3 + x(x^2+2)f_{n+1}^3 + f_n^3 = x(x^2+2)f_{n+3}^3 + (x^2+1)(x^2+2)f_{n+2}^3.$$

Let A, B, and C denote the sets of closed walks of lengths $n+3$, n, and $n-1$, all originating at v_1, respectively. The sum of the weights of all walks in A is f_{n+4}. We define the sum S_1 of the weights of the elements in the product set $A \times A \times A$ as the product of the sums of weights from each component; so $S_1 = f_{n+4}^3$. Correspondingly, the sum of the weights in $B \times B \times B$ equals $S_2 = f_{n+1}^3$, and that in $C \times C \times C$ equals $S_3 = f_n^3$. Then,

$$S_1 + x(x^2+2)S_2 + S_3 = f_{n+4}^3 + x(x^2+2)f_{n+1}^3 + f_n^3.$$

We will now compute the sum $S = S_1 + x(x^2+2)S_2 + S_3$ in a different way. Let (u, v, w) be an arbitrary element of the product set $A \times A \times A$. Table 2 shows the possible cases for such triples and the corresponding sums of weights.

Table 2: Sum of the Weights of the Triples

u begins with a loop?	v begins with a loop?	w begins with a loop?	sum of the weights of triples (u, v, w)
yes	yes	yes	$x^3 f_{n+3}^3$
yes	yes	no	$x^2 f_{n+3}^2 f_{n+2}$
yes	no	yes	$x^2 f_{n+3}^2 f_{n+2}$
yes	no	no	$x f_{n+3} f_{n+2}^2$
no	yes	yes	$x^2 f_{n+3}^2 f_{n+2}$
no	yes	no	$x f_{n+3} f_{n+2}^2$
no	no	yes	$x f_{n+3} f_{n+2}^2$
no	no	no	f_{n+2}^3

It follows from the table that

$$S_1 = x^3 f_{n+3}^3 + 3x^2 f_{n+3}^2 f_{n+2} + 3x f_{n+3} f_{n+2}^2 + f_{n+2}^3.$$

This implies

$$\begin{aligned} S_2 &= x^3 f_n^3 + 3x^2 f_n^2 f_{n-1} + 3x f_n f_{n-1}^2 + f_{n-1}^3; \\ S_3 &= x^3 f_{n-1}^3 + 3x^2 f_{n-1}^2 f_{n-2} + 3x f_{n-1} f_{n-2}^2 + f_{n-2}^3. \end{aligned}$$

Clearly, $S_2 = f_{n+1}^3$ and $S_3 = f_n^3$; so

$$x(x^2+2)S_2 + S_3 = (x^3+2x) f_{n+1}^3 + f_n^3.$$

Now to simplify S_1. First, notice that

$$\begin{aligned} 3x^2 f_{n+3}^2 f_{n+2} &= 2x f_{n+3}^2 (f_{n+3} - f_{n+1}) + x^2 f_{n+2} (x f_{n+2} + f_{n+1})^2 \\ &= 2x f_{n+3}^3 + x^4 f_{n+2}^3 + x^2 f_{n+2}^2 (f_{n+2} - f_n) + x^3 f_{n+2}^2 f_{n+1} \\ &\quad + x^2 f_{n+2} f_{n+1}^2 - 2x f_{n+3}^2 f_{n+1} \\ &= 2x f_{n+3}^3 + (x^4 + x^2) f_{n+2}^3 - x^2 f_{n+2}^2 f_n + x^3 f_{n+2}^2 f_{n+1} \\ &\quad + x^2 f_{n+2} f_{n+1}^2 - 2x f_{n+3}^2 f_{n+1}; \\ 3x f_{n+3} f_{n+2}^2 &= 2x f_{n+2}^2 (x f_{n+2} + f_{n+1}) + x f_{n+3} f_{n+2}^2 \\ &= 2x^2 f_{n+2}^3 + 2 f_{n+2}^2 (f_{n+2} - f_n) + x f_{n+3} f_{n+2}^2 \\ &= (2x^2 + 2) f_{n+2}^3 - 2 f_{n+2}^2 f_n + x f_{n+3} f_{n+2}^2. \end{aligned}$$

Consequently,

$$S_1 = x(x^2+2) f_{n+3}^3 + (x^2+1)(x^2+2) f_{n+2}^3 + F,$$

where

$$F = f_{n+2}^3 - 2x f_{n+3}^2 f_{n+1} + x^3 f_{n+2}^2 f_{n+1} + x^2 f_{n+2} f_{n+1}^2 - x^2 f_{n+2}^2 f_n - 2 f_{n+2}^2 f_n + x f_{n+3} f_{n+2}^2.$$

Since

$$\begin{aligned} -2x f_{n+3}^2 f_{n+1} &= -2x f_{n+1} (x f_{n+2} + f_{n+1})^2 \\ &= -2x^3 f_{n+2}^2 f_{n+1} - 4x^2 f_{n+2} f_{n+1}^2 - 2x f_{n+1}^3, \end{aligned}$$

we have

$$\begin{aligned} F &= f_{n+2}^3 - x^3 f_{n+2}^2 f_{n+1} - 3x^2 f_{n+2} f_{n+1}^2 + x^3 f_{n+2}^2 f_{n+1} - 3x^2 f_{n+2} f_{n+1}^2 \\ &\quad - 2x f_{n+1}^3 - x^2 f_{n+2}^2 f_n - 2f_{n+2}^2 f_n + x f_{n+3} f_{n+2}^2; \\ F + x(x^2+2)S_2 + S_3 &= f_{n+2}^3 - x^3 f_{n+2}^2 f_{n+1} - 3x^2 f_{n+2} f_{n+1}^2 + x^3 f_{n+1}^3 - x^2 f_{n+2}^2 f_n \\ &\quad - 2f_{n+2}^2 f_n + x f_{n+3} f_{n+2}^2 + f_n^3. \end{aligned}$$

Using the identities

$$\begin{aligned} x^3 f_{n+2}^2 f_{n+1} &= x^2 f_{n+2}^2 (f_{n+2} - f_n) \\ &= x^2 f_{n+2}^3 - x^2 f_{n+2}^2 f_n; \\ x^2 f_{n+2} f_{n+1}^2 &= f_{n+2}(f_{n+2} - f_n)^2 \\ &= f_{n+2}^3 - 2f_{n+2}^2 f_n + f_{n+2} f_n^2; \\ x^2 f_{n+2} f_{n+1}^2 &= x^3 f_{n+1}^3 + x^2 f_{n+1}^2 f_n, \end{aligned}$$

we have

$$\begin{aligned} F + x(x^2+2)S_2 + S_3 &= -x^2 f_{n+2}^3 - f_{n+2} f_n^2 - x^2 f_{n+1}^2 f_n - x^2 f_{n+2} f_{n+1}^2 + x f_{n+3} f_{n+2}^2 + f_n^3 \\ &= x f_{n+2}^2 (f_{n+3} - x f_{n+2}) - f_{n+2} f_n^2 - x^2 f_{n+1}^2 f_n - x^2 f_{n+2} f_{n+1}^2 + f_n^3 \\ &= x f_{n+2} f_{n+1} (f_{n+2} - x f_{n+1}) - f_{n+2} f_n^2 - x^2 f_{n+1}^2 f_n + f_n^3 \\ &= x f_{n+1} f_n (f_{n+2} - x f_{n+1}) - f_{n+2} f_n^2 + f_n^3 \\ &= x f_{n+1} f_n^2 - f_{n+2} f_n^2 + f_n^3 \\ &= -f_n^2 (f_{n+2} - x f_{n+1}) + f_n^3 \\ &= -f_n^3 + f_n^3 \\ &= 0. \end{aligned}$$

Consequently,

$$S = x(x^2+2) f_{n+3}^3 + (x^2+1)(x^2+2) f_{n+2}^3,$$

as expected. □

Part II. To establish the identity with $g_n = l_n$, we will confirm its equivalent form:

$$l_{n+4}^3 + x(x^2+2) l_{n+1}^3 + l_n^3 = x(x^2+2) l_{n+3}^3 + (x^2+1)(x^2+2) l_{n+2}^3.$$

Let A, B, and C denote the sets of closed walks of lengths $n+4$, $n+1$, and n in D_1, respectively. The sum of the weights of all walks in A is l_{n+4}. We define the sum S_1 of the weights of the elements in the product set $A \times A \times A$ as the product of the sums of weights from each component; so, the sum S_1 of the weights of the elements in $A \times A \times A$ equals $S_1 = l_{n+4}^3$. Correspondingly, the sum of the weights S_2 of the elements in $B \times B \times B$ equals $S_2 = l_{n+1}^3$, and the sum of the weights S_3 of the elements in $C \times C \times C$ equals $S_3 = l_n^3$. Then,

$$S_1 + x(x^2+2)S_2 + S_3 = l_{n+4}^3 + x(x^2+2) l_{n+1}^3 + l_n^3.$$

It now remains to show that

$$S_1 + x(x^2+2)S_2 + S_3 = x(x^2+2) l_{n+3}^3 + (x^2+1)(x^2+2) l_{n+2}^3.$$

Since the sum of the weights of all closed walks in A equals $f_{n+5} + f_{n+3} = l_{n+4}$, we have

$$\begin{aligned} S_1 &= (x l_{n+3} + l_{n+2})^3 \\ &= x^3 l_{n+3}^3 + 3x^2 l_{n+3}^2 l_{n+2} + 3x l_{n+3} l_{n+2}^2 + l_{n+2}^3. \end{aligned}$$

Replacing f_n with l_n in the second half of the proof in Part 1, it follows that

$$S_1 = x(x^2+2)l_{n+3}^3 + (x^2+1)(x^2+2)l_{n+2}^3 + G,$$

where

$$\begin{aligned} G &= l_{n+2}^3 - 2xl_{n+3}^2 l_{n+1} + x^3 l_{n+2}^2 l_{n+1} + x^2 l_{n+2} l_{n+1}^2 - x^2 l_{n+2}^2 l_n - 2l_{n+2}^2 l_n + xl_{n+3} l_{n+2}^2 \\ &= l_{n+2}^3 - x^3 l_{n+2}^2 l_{n+1} - 3x^2 l_{n+2} l_{n+1}^2 + x^3 l_{n+2}^2 l_{n+1} - 3x^2 l_{n+2} l_{n+1}^2 \\ &\quad - 2xl_{n+1}^3 - x^2 l_{n+2}^2 l_n - 2l_{n+2}^2 l_n + xl_{n+3} l_{n+2}^2. \end{aligned}$$

Since $S_2 = (f_{n+2} + f_n)^3 = l_{n+1}^3$ and hence $S_3 = l_n^3$, it also follows that

$$\begin{aligned} G + x(x^2+2)S_2 + S_3 &= l_{n+2}^3 - x^3 l_{n+2}^2 l_{n+1} - 3x^2 l_{n+2} l_{n+1}^2 + x^3 l_{n+1}^3 - x^2 l_{n+2}^2 l_n \\ &\quad - 2l_{n+2}^2 l_n + xl_{n+3} l_{n+2}^2 + l_n^3 \\ &= -l_n^3 + l_n^3 \\ &= 0. \end{aligned}$$

Thus,

$$S = x(x^2+2)l_{n+3}^3 + (x^2+1)(x^2+2)l_{n+2}^3,$$

as desired. □

4. Jacobsthal Implications

Using the gibonacci-Jacobsthal relationships in Section 1, we now find the Jacobsthal counterpart of identity (2.1). Suppose $g_n = f_n$. Replacing x with $u = 1/\sqrt{x}$, equation (2.1) yields

$$x^2\sqrt{x} f_{n+4}^3 = x(2x+1)f_{n+3}^3 + (x+1)(2x+1)\sqrt{x} f_{n+2}^3 - x(2x+1)f_{n+1}^3 - x^2\sqrt{x} f_n^3,$$

where $f_n = f_n(u)$. Multiplying both sides with $x^{3(n+3)/2}$, we get

$$J_{n+4}^3(x) = (2x+1)J_{n+3}^3(x) + x(x+1)(2x+1)J_{n+2}^3(x) - (2x+1)x^3 J_{n+1}^3(x) - x^6 J_n^3(x).$$

When $g_n = l_n$, likewise we get

$$j_{n+4}^3(x) = (2x+1)j_{n+3}^3(x) + x(x+1)(2x+1)j_{n+2}^3(x) - (2x+1)x^3 j_{n+1}^3(x) - x^6 j_n^3(x).$$

Combining the two cases, we get

$$c_{n+4}^3 = (2x+1)c_{n+3}^3 + x(x+1)(2x+1)c_{n+2}^3 - (2x+1)x^3 c_{n+1}^3 - x^6 c_n^3. \tag{4.1}$$

In particular, we have

$$C_{n+4}^3 = 5C_{n+3}^3 + 30C_{n+2}^3 - 40C_{n+1}^3 - 64C_n^3.$$

4.1. **Graph-theoretic Model.** Next, we construct a graph-theoretic model for the Jacobsthal identity (4.1). We accomplish this using the weighted digraph D_2 in Figure 2 [9]. Using its weighted adjacency matrix

$$M = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix},$$

it follows by induction that

$$M^n = \begin{bmatrix} J_{n+1}(x) & xJ_n(x) \\ J_n(x) & xJ_{n-1}(x) \end{bmatrix},$$

where $n \geq 1$.

Consequently, the sum of the weights of closed walks of length n originating at v_1 is $J_{n+1}(x)$, and that of those originating at v_2 is $xJ_{n-1}(x)$. So, the sum of the weights of all closed walks

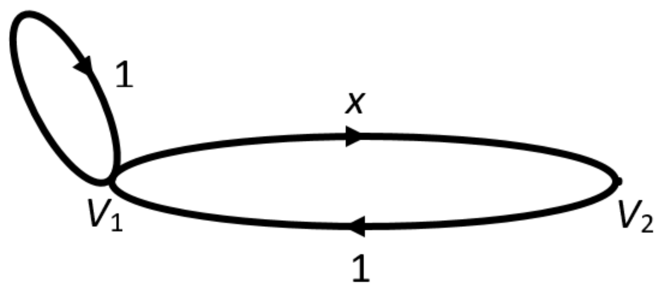

FIGURE 2. Weighted Digraph D_2

of length n in the digraph is $J_{n+1}(x) + xJ_{n-1}(x) = j_n(x)$. These facts play a pivotal role in the pursuit of the graph-theoretic model.

Part I. Let $c_n = J_n(x)$. We will then confirm the following equivalent form:

$$J^3_{n+4}(x) + (2x + 1)x^3 J^3_{n+1}(x) + x^6 J^3_n(x) = (2x + 1)J^3_{n+3}(x) + x(x + 1)(2x + 1)J^3_{n+2}(x).$$

Let A, B, and C denote the sets of closed walks of lengths $n + 3$, n, and $n - 1$ originating at v_1, respectively. The sums of the weights of such walks are $J_{n+4}(x)$, $J_{n+1}(x)$, and $J_n(x)$, respectively. We define the sum S_1 of the weights of the elements in the product set $A \times A \times A$ as the product of the sums of weights from each component; so $S_1 = J^3_{n+4}(x)$. Correspondingly, the sum of the weights S_2 of the elements in $B \times B \times B$ equals $S_2 = J^3_{n+1}(x)$, and the sum of the weights S_3 of the elements in $C \times C \times C$ equals $S_3 = J^3_n(x)$. Then, the desired sum S is given by

$$\begin{aligned} S &= S_1 + (2x + 1)x^3 S_2 + x^6 S_3 \\ &= J^3_{n+4}(x) + (2x + 1)x^3 J^3_{n+1}(x) + x^6 J^3_n(x). \end{aligned}$$

To compute the sum S in a different way, consider an arbitrary element (u, v, w) of the product $A \times A \times A$. Table 3 shows the various cases for the triples and their corresponding weights. It follows from the table that the total contribution S_1 from such triples is given by

$$S_1 = J^3_{n+3}(x) + 3xJ^2_{n+3}(x)J_{n+2}(x) + 3x^2 J_{n+3}(x)J^2_{n+2}(x) + x^3 J^3_{n+2}(x).$$

Table 3: Sum of the Weights of the Triples

u begins with a loop?	v begins with a loop?	w begins with a loop?	sum of the weights of triples (u, v, w)
yes	yes	yes	$x^3 J^3_{n+3}(x)$
yes	yes	no	$xJ^2_{n+3}(x)J_{n+2}(x)$
yes	no	yes	$xJ^2_{n+3}(x)J_{n+2}(x)$
yes	no	no	$x^2 J_{n+3}(x)J^2_{n+2}(x)$
no	yes	yes	$xJ^2_{n+3}(x)J_{n+2}(x)$
no	yes	no	$x^2 J_{n+3}(x)J^2_{n+2}(x)$
no	no	yes	$x^2 J_{n+3}(x)J^2_{n+2}(x)$
no	no	no	$x^3 J^3_{n+2}(x)$

It then follows that

$$\begin{aligned} S_2 &= J_n^3(x) + 3xJ_n^2(x)J_{n-1}(x) + 3x^2J_n(x)J_{n-1}^2(x) + x^3J_{n-1}^3(x) \\ &= J_{n+1}^3(x); \\ S_3 &= J_{n-1}^3(x) + 3xJ_{n-1}^2(x)J_{n-2}(x) + 3x^2J_{n-1}(x)J_{n-2}^2(x) + x^3J_{n-2}^3(x) \\ &= J_n^3(x). \end{aligned}$$

In the rest of the section, we *omit* the argument in the functional notation for the sake of brevity and clarity. We will now show that

$$S_1 + (2x+1)x^3S_2 + x^6S_3 = (2x+1)J_{n+3}^3 + x(x+1)(2x+1)J_{n+2}^3.$$

To rewrite S_1 in a different form, first notice that

$$\begin{aligned} 3xJ_{n+3}^2J_{n+2} &= 2xJ_{n+3}^2J_{n+2} + xJ_{n+3}^2J_{n+2} \\ &= 2xJ_{n+3}^2(J_{n+3} - xJ_{n+1}) + xJ_{n+2}^2(J_{n+2} + xJ_{n+1})^2 \\ &= 2xJ_{n+3}^3 + xJ_{n+2}^3 - 2x^2J_{n+3}^3J_{n+1} + 2x^2J_{n+2}^2J_{n+1} + x^3J_{n+2}J_{n+1}^2; \\ 3x^2J_{n+3}J_{n+2}^2 &= 3x^2J_{n+2}^2(J_{n+2} + xJ_{n+1}) \\ &= 3x^2J_{n+2}^3 + x^3J_{n+2}^2(J_{n+2} - xJ_n) + 2x^3J_{n+2}^2J_{n+1} \\ &= (x^3 + 3x^2)J_{n+2}^3 - x^4J_{n+2}^2J_n + 2x^3J_{n+2}^2J_{n+1}. \end{aligned}$$

We then have

$$S_1 = (2x+1)J_{n+3}^3 + x(x+1)(2x+1)J_{n+2}^3 + H + I + J + K,$$

where

$$\begin{aligned} H &= -2x^2J_{n+3}^2J_{n+1} + 2x^2J_{n+2}^2J_{n+1} \\ &= -2x^2J_{n+1}(J_{n+2} + xJ_{n+1})^2 + 2x^2J_{n+2}^2J_{n+1} \\ &= -2x^4J_{n+1}^3 - 2x^3J_{n+2}J_{n+1}^2 - 2x^3J_{n+2}J_{n+1}^2 \\ &= -2x^4J_{n+1}^3 - 2x^3J_{n+1}^2(J_{n+1} + xJ_n) - 2x^3J_{n+2}J_{n+1}^2 \\ &= -2x^4J_{n+1}^3 - 2x^3J_{n+1}^3 - 2x^4J_{n+1}^2J_n - 2x^3J_{n+2}J_{n+1}^2; \\ I &= x^3J_{n+2}J_{n+1}^2 \\ &= x^3J_{n+1}^3 + x^4J_{n+1}^2J_n; \\ J &= -x^4J_{n+2}^2J_n \\ &= -x^4J_{n+1}^2J_n - x^6J_n^3 - 2x^5J_{n+1}J_n^2; \\ K &= 2x^3J_{n+2}^3J_{n+1} \\ &= 2x^3J_{n+1}^3 + 4x^4J_{n+1}^2J_n + 2x^5J_{n+1}J_n^2. \end{aligned}$$

Then,

$$H + I + J + K = -(2x^4 + x^3)J_{n+1}^3 + 2x^3J_{n+1}^3 + 2x^4J_{n+1}^2J_n - x^6J_n^3 - 2x^3J_{n+2}J_{n+1}^2.$$

Consequently,

$$\begin{aligned} H + I + J + K + (2x+1)x^3S_2 + x^6S_3 &= 2x^3J_{n+1}^3 + 2x^4J_{n+1}^2J_n - 2x^3J_{n+2}J_{n+1}^2 \\ &= 2x^3J_{n+1}^2(J_{n+1} + xJ_n) - 2x^3J_{n+2}J_{n+1}^2 \\ &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} S &= S_1 + (2x+1)x^3 S_2 + x^6 S_3 \\ &= (2x+1)J_{n+3}^3 + x(x+1)(2x+1)J_{n+2}^3, \end{aligned}$$

as desired. □

Part II. Suppose $c_n = j_n$. We will then confirm that

$$j_{n+4}^3 + (2x+1)x^3 j_{n+1}^3 + x^6 j_n^3 = (2x+1)j_{n+3}^3 + x(x+1)(2x+1)j_{n+2}^3.$$

This time, we focus on all closed walks of lengths $n+4$, $n+1$, and n. Let A, B, and C denote the sets of closed walks of lengths $n+4$, $n+1$, and n, all originating at v_1; and R, S, and T the sets of those originating at v_2. The sum of the weights of all closed walks of length $n+4$ is j_{n+4}; so we define the sum S_1 of the weights of the elements in the product set $E \times E \times E$ is j_{n+4}^3, where $E = A \cup R$. Likewise, the sum S_2 of the weights of the elements in $F \times F \times F$ is j_{n+1}^3, where $F = B \cup S$; and the sum S_3 of those in $G \times G \times G$ is j_n^3, where $G = C \cup T$.

Thus, the desired sum S on the left side of the identity is given by

$$\begin{aligned} S &= S_1 + (2x+1)x^3 S_2 + x^6 S_3 \\ &= j_{n+4}^3 + (2x+1)x^3 j_{n+1}^3 + x^6 j_n^3. \end{aligned}$$

It now suffices to show that

$$S_1 + (2x+1)x^3 S_2 + x^6 S_3 = (2x+1)j_{n+3}^3 + x(x+1)(2x+1)j_{n+2}^3.$$

This can be achieved by employing a technique similar to the one used in the graph-theoretic proof of identity (2.1) with $g_n = l_n$. In the interest of brevity, we omit the details.

Finally, we add that using the relationships in Table 1, identity (2.1) can be extended to Vieta and Chebyshev polynomials.

5. Acknowledgment

The author thanks the reviewer for thoughtful comments and suggestions, which helped improve the quality of the original version.

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MSC2010: 11B37, 11B39, 33B50

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