GENERALIZATIONS OF HERMITE'S IDENTITY AND APPLICATIONS

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Abstract. Hermite's identity states that

$$\sum_{0\le k\le n-1}\left\lfloor x+\frac{k}{n}\right\rfloor = \lfloor nx\rfloor \text{ for all } x\in\mathbb{R} \text{ and } n\in\mathbb{N}.$$

In this article, we give a generalization of this identity and show some applications. For example, we consider the above sum when k ranges over the integers from a to b, where $a<b$ are integers. Then, we apply it to give another proof of a recent result of Tverberg. We also obtain a formula for the corresponding sum when k ranges over a complete residue system modulo n.

1. Introduction

For each $x\in\mathbb{R}$, the greatest integer function of x (or the floor function of x), denoted by $\lfloor x\rfloor$, is defined as the largest integer not exceeding x, and the fractional part of x, denoted by $\{x\}$, is given by $\{x\}=x-\lfloor x\rfloor$.

The study of sums involving the floor function and the distribution of fractional parts of certain sequences has been a popular area of research. For example, Dirichlet's divisor problem is to obtain the best estimate for the sum

$$\sum_{n=1}^{N} d(n) = \sum_{m=1}^{N}\left\lfloor \frac{N}{m}\right\rfloor,$$

or more precisely, the infimum of $\theta>0$ such that

$$\left|\sum_{m\le N}\left\lfloor\frac{N}{m}\right\rfloor-(N\log N+(2\gamma-1)N)\right|\ll N^{\theta},$$

where $d(n)$ is the number of positive divisors of n and γ is Euler's constant. For some recent results on Dirichlet's divisor problem, we refer the reader to Khan [8]; Liu, Shparlinski, and Zhang [11]; and Pongsriiam and Vaughan [20, 21]. Other sums involving the floor function are also considered by Jacobsthal [7], Carlitz [1], Grimson [6], and recently by Tverberg [23], Onphaeng and Pongsriiam [13], and Thanatipanonda and Wong [22]. For an elementary sum, we recall Hermite's identity that states

$$\sum_{0\le k\le n-1}\left\lfloor x+\frac{k}{n}\right\rfloor = \lfloor nx\rfloor \quad \text{for all } x\in\mathbb{R} \text{ and } n\in\mathbb{N}.$$

In this article, we generalize Hermite's identity and give some applications. For example, in Corollary 3.3, we obtain a closed form for a sum similar to Hermite's identity, where $a<b$ are integers and k ranges from a to b. Then, we apply this result to give another proof of a recent result of Tverberg. Theorem 3.6 presents a formula for the corresponding sum when k ranges over a complete residue system modulo n.

In Section 2, we recall some preliminaries and lemmas for the reader's convenience. Then, we give our main results and their applications in Section 3. For other topics related to the floor function, see for example, [2, 3, 4, 9, 14, 15, 16, 18, 17, 19].

2. Preliminaries and Lemmas

We first recall that for integers $a < b$, the notations such as

$$\sum_{n=a}^{b} f(n), \quad \sum_{a\leq n\leq b} f(n), \quad \sum_{a\leq n<b+1} f(n), \text{ and } \quad f(a)+f(a+1)+\cdots+f(b)$$

have the same meaning. In addition, we assign the value zero to an empty sum such as $\sum_{0\leq n\leq -1} f(n)$. Next, we recall some basic properties of $\lfloor x\rfloor$ and $\{x\}$, which are used throughout this article, sometimes without reference.

Lemma 2.1. *Let $x, y \in \mathbb{R}$ and $n \in \mathbb{Z}$. Then, the following statements hold.*

(i) $\lfloor x\rfloor = n$ *if and only if* $n \leq x < n+1$.
(ii) *If* $x < y$, *then* $\lfloor x\rfloor \leq \lfloor y\rfloor$.
(iii) $\lfloor x+n\rfloor = \lfloor x\rfloor + n$.
(iv) *If* $x \in \mathbb{Z}$, *then* $\lfloor x\rfloor = x$.
(v) $0 \leq \{x\} < 1$.

Proof. These are well-known and can be proved easily. For more details, we refer the reader to Chapter 3 of the book by Graham, Knuth, and Patashnik [5]. □

We will use Fibonacci numbers in examples. Recall that the Fibonacci sequence $(F_n)_{n\geq 0}$ is defined by the recurrence relation $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. We have the following result.

Lemma 2.2. *Let $\alpha = \frac{1+\sqrt{5}}{2}$ be the golden ratio and $\beta = \frac{1-\sqrt{5}}{2}$ its conjugate. Then, the following statements hold.*

(i) *(Binet's formula)* $F_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$ *for all* $n \geq 0$.
(ii) $F_n\alpha = F_{n+1} - \beta^n$ *for all* $n \geq 0$.

Proof. Statement (i) is well-known and can be found in the book by Koshy [10, p. 78]. For (ii), we first observe that $\alpha\beta = -1$ and $\beta^2 + 1 = \beta + 2 = -\sqrt{5}\beta$. Then, by Binet's formula, $F_n\alpha$ is equal to

$$\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)\alpha = \frac{\alpha^{n+1}+\beta^{n-1}}{\alpha-\beta} = \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} + \frac{\beta^{n+1}+\beta^{n-1}}{\alpha-\beta} = F_{n+1} + \frac{\beta^{n+1}+\beta^{n-1}}{\alpha-\beta}.$$

The second term above is

$$\frac{\beta^{n-1}(\beta^2+1)}{\alpha-\beta} = \frac{\beta^{n-1}(-\sqrt{5}\beta)}{\sqrt{5}} = -\beta^n.$$

This proves (ii). □

3. Main Results

We begin with a simple generalization of Hermite's identity.

Theorem 3.1. *Let $x \in \mathbb{R}$, $m, n \in \mathbb{Z}$, and $0 \le m \le n$. Then,*

$$\sum_{0 \le k \le m-1} \left\lfloor x + \frac{k}{n} \right\rfloor = \max\left\{m\lfloor x\rfloor, m\lfloor x\rfloor + m - n + \lfloor n\{x\}\rfloor\right\}. \tag{3.1}$$

In particular, if $m = n$ are positive integers, then (3.1) is the same as Hermite's identity.

Proof. By Lemma 2.1(v), we know that $n\{x\} < n$, and so $-n + \lfloor n\{x\}\rfloor < 0$. When $m = 0$, the left side of (3.1) is an empty sum and so it is equal to 0, and the right side of (3.1) is also equal to $\max\{0, -n + \lfloor n\{x\}\rfloor\} = 0$. Therefore, we may assume that $m \ge 1$. Then, by the definition of $\{x\}$ and by Lemma 2.1(iii), we have

$$\left\lfloor x + \frac{k}{n} \right\rfloor = \left\lfloor \lfloor x\rfloor + \{x\} + \frac{k}{n} \right\rfloor = \lfloor x\rfloor + \left\lfloor \frac{n\{x\} + k}{n} \right\rfloor.$$

So, the left side of (3.1) is equal to

$$m\lfloor x\rfloor + \sum_{0 \le k \le m-1} \left\lfloor \frac{n\{x\} + k}{n} \right\rfloor.$$

Since $0 \le n\{x\} < n$, there exists an $r \in \{0, 1, 2, \dots, n-1\}$ such that $r \le n\{x\} < r + 1$. By Lemma 2.1(i), we have $r = \lfloor n\{x\}\rfloor$. For $0 \le k \le m - 1$, we have

$$0 \le \frac{k}{n} \le \frac{n\{x\} + k}{n} < \frac{n + k}{n} < 2.$$

So, $\left\lfloor \frac{n\{x\}+k}{n} \right\rfloor = 0$ or 1. In addition, $\left\lfloor \frac{n\{x\}+k}{n} \right\rfloor = 1 \Leftrightarrow \frac{n\{x\}+k}{n} \ge 1$. If $k \ge n - r$, then $\frac{n\{x\}+k}{n} \ge \frac{n+n\{x\}-r}{n} \ge 1$. If $k < n - r$, then $\frac{n\{x\}+k}{n} \le \frac{n\{x\}+n-r-1}{n} < 1$. Therefore, $\left\lfloor \frac{n\{x\}+k}{n} \right\rfloor = 1$ if and only if $k \ge n - r$. Hence,

$$\sum_{0 \le k \le m-1} \left\lfloor \frac{n\{x\} + k}{n} \right\rfloor = \sum_{n-r \le k \le m-1} 1 = \begin{cases} m - n + r, & \text{if } m \ge n - r + 1; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the left side of (3.1) is equal to $m\lfloor x\rfloor + m - n + r$ if $m - n + r \ge 1$, and is equal to $m\lfloor x\rfloor$ if $m - n + r \le 0$. Recalling that $r = \lfloor n\{x\}\rfloor$, the result follows. □

We can extend Theorem 3.1 as follows.

Theorem 3.2. *Let $x \in \mathbb{R}$ and $m, n \in \mathbb{N}$. Then,*

$$\sum_{0 \le k \le m-1} \left\lfloor x + \frac{k}{n} \right\rfloor = \frac{n}{2} \left\lfloor \frac{m}{n} \right\rfloor \left(\left\lfloor \frac{m}{n} \right\rfloor - 1 \right) + \left\lfloor \frac{m}{n} \right\rfloor \lfloor nx\rfloor + \left\lfloor \frac{m}{n} \right\rfloor r$$
$$+ \max\left\{r\lfloor x\rfloor, r\lfloor x\rfloor + r - n + \lfloor n\{x\}\rfloor\right\}, \tag{3.2}$$

where $r = m - n\left\lfloor \frac{m}{n} \right\rfloor$ is the remainder in the division of m by n.

Proof. If $m < n$, then $\left\lfloor \frac{m}{n} \right\rfloor = 0$, $r = m$, and the right side of (3.2) is the same as that of (3.1). If $m = n$, then $r = 0$ and the right side of (3.2) is equal to $\lfloor nx\rfloor + \max\{0, -n + \lfloor n\{x\}\rfloor\} = \lfloor nx\rfloor$. This shows that if $m \le n$, then (3.2) reduces to (3.1). Therefore, we assume that $m > n$. By the division algorithm, there exist $q \in \mathbb{N}$ and $r \in \mathbb{Z}$ such that $m = nq + r$ and $0 \le r < n$. Then, $\left\lfloor \frac{m}{n} \right\rfloor = \left\lfloor q + \frac{r}{n} \right\rfloor = q + \left\lfloor \frac{r}{n} \right\rfloor = q$ and $r = m - nq = m - n\left\lfloor \frac{m}{n} \right\rfloor$. Thus, we have

$$\sum_{0 \le k \le m-1} \left\lfloor x + \frac{k}{n} \right\rfloor = \sum_{\ell=0}^{q-1} \sum_{\ell n \le k \le (\ell+1)n-1} \left\lfloor x + \frac{k}{n} \right\rfloor + \sum_{qn \le k \le qn+r-1} \left\lfloor x + \frac{k}{n} \right\rfloor.$$

By replacing k by $k + \ell n$ in the first sum and k by $k + qn$ in the second sum on the right side of the above equation, we see that

$$\sum_{0\le k\le m-1} \left\lfloor x + \frac{k}{n} \right\rfloor = \sum_{\ell=0}^{q-1} \sum_{0\le k\le n-1} \left\lfloor x + \frac{k+\ell n}{n} \right\rfloor + \sum_{0\le k\le r-1} \left\lfloor x + \frac{k+qn}{n} \right\rfloor . \tag{3.3}$$

Since $\left\lfloor x + \frac{k+\ell n}{n} \right\rfloor = \left\lfloor x + \frac{k}{n} + \ell \right\rfloor = \ell + \left\lfloor x + \frac{k}{n} \right\rfloor$, we obtain, by Theorem 3.1, that the first sum on the right side of (3.3) is

$$\sum_{\ell=0}^{q-1} \left(n\ell + \sum_{0\le k\le n-1} \left\lfloor x + \frac{k}{n} \right\rfloor \right) = \sum_{\ell=0}^{q-1} (n\ell + \lfloor nx \rfloor) = \frac{nq(q-1)}{2} + q \lfloor nx \rfloor . \tag{3.4}$$

The second sum can also be evaluated by Theorem 3.1 and is equal to

$$qr + \max \{ r \lfloor x \rfloor , r \lfloor x \rfloor + r - n + \lfloor n\{x\} \rfloor \} . \tag{3.5}$$

Recall that $q = \left\lfloor \frac{m}{n} \right\rfloor$ and $r = m - n \left\lfloor \frac{m}{n} \right\rfloor$ is the remainder in the division of m by n. Then, substituting (3.4) and (3.5) in (3.3) leads to the desired result. □

Now, we can extend Hermite's identity to a sum where k ranges over the integers from a to b for any $a < b$ as follows.

Corollary 3.3. *Let $x \in \mathbb{R}$, $n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, and $a < b$. Then,*

$$\sum_{a\le k<b} \left\lfloor x + \frac{k}{n} \right\rfloor = \frac{n}{2} \left\lfloor \frac{m}{n} \right\rfloor \left(\left\lfloor \frac{m}{n} \right\rfloor - 1 \right) + \left\lfloor \frac{m}{n} \right\rfloor r + \left\lfloor \frac{m}{n} \right\rfloor \lfloor nx \rfloor$$
$$+ \quad \max \left\{ r \left\lfloor x + \frac{a}{n} \right\rfloor , r \left\lfloor x + \frac{a}{n} \right\rfloor + r - n + \left\lfloor n \left\{ x + \frac{a}{n} \right\} \right\rfloor \right\} ,$$

where $m = b - a$ and r is the remainder in the division of $b - a$ by n.

Proof. Replacing k by $k + a$, we see that the left side is

$$\sum_{0\le k<b-a} \left\lfloor \left(x + \frac{a}{n} \right) + \frac{k}{n} \right\rfloor ,$$

which is suitable for an application of Theorem 3.2. This leads to the desired result. □

For each $a, b, k \in \mathbb{Z}$ and $m \in \mathbb{N}$, define $f_{a,b,m}(k)$ by

$$f_{a,b,m}(k) = \left\lfloor \frac{a+b+k}{m} \right\rfloor - \left\lfloor \frac{a+k}{m} \right\rfloor - \left\lfloor \frac{b+k}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor .$$

For $N \in \mathbb{N}$, Jacobsthal [7] introduced the sum

$$S_{a,b,m}(N) = \sum_{k=0}^{N} f_{a,b,m}(k),$$

and used the analytic method to show that $S_{a,b,m}(N) \ge 0$ for all $N \ge 1$. Tverberg [23] gives another proof of this inequality, generalizes the sum $S_{a,b,m}(N)$, and obtains some upper and lower bounds for those generalizations. Onphaeng and Pongsriiam [13] and Thanatipanonda and Wong [22] extend Tverberg's result further; we refer the reader to [13] or [22] for some open problems concerning the bounds for Tverberg's generalization of the sum $S_{a,b,m}(N)$. In the next corollary, we combine Tverberg's argument and our result to give another proof of the inequality

$$S_{a,b,m}(N) \ge 0 \quad \text{for all } N \ge 1. \tag{3.6}$$

Corollary 3.4. *The inequality* (3.6) *holds.*

Proof. Tverberg observes that $f_{a,b,m}(k)$ is invariant when we replace a, b, k by $a \pm m$, $b \pm m$, $k \pm m$, respectively. So, we can assume that $0 \le a, b < m$. Since the sum $S_{a,b,m}(N)$ is taken over $k = 0, 1, \dots, N$, it suffices to consider only the case $1 \le N \le m-1$. By the definition of $S_{a,b,m}(N)$, we have $S_{a,b,m}(N) = A_1 - A_2 - A_3 + A_4$, where

$$A_1 = \sum_{0 \le k \le N} \left\lfloor \frac{a+b}{m} + \frac{k}{m} \right\rfloor, \quad A_2 = \sum_{0 \le k \le N} \left\lfloor \frac{a}{m} + \frac{k}{m} \right\rfloor,$$
$$A_3 = \sum_{0 \le k \le N} \left\lfloor \frac{b}{m} + \frac{k}{m} \right\rfloor, \text{ and } \quad A_4 = \sum_{0 \le k \le N} \left\lfloor \frac{k}{m} \right\rfloor.$$

By Theorem 3.1, A_1 is equal to

$$\max\left\{(N+1)\left\lfloor \frac{a+b}{m} \right\rfloor, (N+1)\left\lfloor \frac{a+b}{m} \right\rfloor + \alpha_1\right\} = (N+1)\left\lfloor \frac{a+b}{m} \right\rfloor + \max\{0, \alpha_1\},$$

where $\alpha_1 = N+1-m+\left\lfloor m\left\{\frac{a+b}{m}\right\}\right\rfloor$. Since $m\left\{\frac{a+b}{m}\right\} = m\left(\frac{a+b}{m} - \left\lfloor \frac{a+b}{m} \right\rfloor\right) = a+b-m\left\lfloor \frac{a+b}{m} \right\rfloor \in \mathbb{Z}$, we see that $\alpha_1 = N+1-m+a+b-m\left\lfloor \frac{a+b}{m} \right\rfloor$. Similarly, since $0 \le a, b < m$, we obtain $\left\lfloor \frac{a}{m} \right\rfloor = \left\lfloor \frac{b}{m} \right\rfloor = 0$ and $A_2 = \max\{0, \alpha_2\}$, $A_3 = \max\{0, \alpha_3\}$, where $\alpha_2 = N+1-m+a$, $\alpha_3 = N+1-m+b$. In addition, since $N < m$, $A_4 = 0$. Therefore,

$$S_{a,b,m}(N) = (N+1)\left\lfloor \frac{a+b}{m} \right\rfloor + \max\{0, \alpha_1\} - \max\{0, \alpha_2\} - \max\{0, \alpha_3\}. \tag{3.7}$$

Since $0 \le a+b < 2m$, $\left\lfloor \frac{a+b}{m} \right\rfloor = 0$ or 1. So, we divide our calculation into two cases.

Case 1. $0 \le a+b < m$. Then, $\left\lfloor \frac{a+b}{m} \right\rfloor = 0$, $\alpha_1 = N+1-m+a+b$, $\alpha_1 \ge \alpha_2$, $\alpha_1 \ge \alpha_3$, and (3.7) becomes

$$S_{a,b,m}(N) = \max\{0, \alpha_1\} - \max\{0, \alpha_2\} - \max\{0, \alpha_3\}. \tag{3.8}$$

If $\alpha_1 < 0$, then α_2 and $\alpha_3 < 0$ and so, the right side of (3.8) is zero. Suppose $\alpha_1 \ge 0$. Then, there are four possible values for the right side of (3.8), namely, α_1, $\alpha_1 - \alpha_2$, $\alpha_1 - \alpha_3$, or $\alpha_1 - \alpha_2 - \alpha_3$. They satisfy $\alpha_1 = \alpha_1$, $\alpha_1 - \alpha_2 = (N+1-m+a+b) - (N+1-m+a) = b$, $\alpha_1 - \alpha_3 = (N+1-m+a+b) - (N+1-m+b) = a$, and $\alpha_1 - \alpha_2 - \alpha_3 = m-1-N$, which are nonnegative.

Case 2. $m \le a+b < 2m$. We have $\left\lfloor \frac{a+b}{m} \right\rfloor = 1$, $\alpha_1 = N+1-m+a+b-m$, $\alpha_1 < \alpha_2$, $\alpha_1 < \alpha_3$, and (3.7) becomes

$$S_{a,b,m}(N) = (N+1) + \max\{0, \alpha_1\} - \max\{0, \alpha_2\} - \max\{0, \alpha_3\}. \tag{3.9}$$

If $\alpha_1 \ge 0$, then $\alpha_2, \alpha_3 \ge 0$ and the right side of (3.9) is

$$N+1+\alpha_1-\alpha_2-\alpha_3 = N+1+(N+1-2m+a+b)-(N+1-m+a)-(N+1-m+b) = 0.$$

If $\alpha_1 < 0$, then the right side of (3.9) is $N+1$, $N+1-\alpha_2$, $N+1-\alpha_3$, or $N+1-\alpha_2-\alpha_3 = N+1$, $m-a$, $m-b$, or $-\alpha_1$, which are nonnegative. Therefore, $S_{a,b,m}(N) \ge 0$, as required. This completes the proof. □

Example 3.5. *Let* $(F_n)_{n\ge 0}$ *be the Fibonacci sequence,* $\alpha = \frac{1+\sqrt{5}}{2}$ *the golden ratio, and* $3 \le m \le n$. *Then,*

$$\sum_{0 \le k \le F_m - 1} \left\lfloor \alpha + \frac{k}{F_n} \right\rfloor = \begin{cases} 2F_m - F_{n-2}, & \text{if } m \ge n-2; \\ F_m, & \text{otherwise.} \end{cases}$$

Proof. By writing $n\{x\} = nx - n\lfloor x \rfloor$, we see that $\lfloor n\{x\} \rfloor = \lfloor nx \rfloor - n\lfloor x \rfloor$. The expression $m\lfloor x \rfloor + m - n + \lfloor n\{x\} \rfloor$ in Theorem 3.1 can be written as $\lfloor nx \rfloor - (n-m)(\lfloor x \rfloor + 1)$. Then, the sum in question is equal to

$$\max\{F_m \lfloor \alpha \rfloor, \lfloor F_n\alpha \rfloor - (F_n - F_m)(\lfloor \alpha \rfloor + 1)\} = \max\{F_m, \lfloor F_n\alpha \rfloor - 2(F_n - F_m)\}. \tag{3.10}$$

Let $\beta = \frac{1-\sqrt{5}}{2}$. Since $-1 < \beta < 0$, we obtain that $0 < \beta^n < 1$ if n is even and $-1 < \beta^n < 0$ if n is odd. Therefore, we obtain, by Lemma 2.2(ii), that $F_{n+1} < F_n\alpha < F_{n+1} + 1$ if n is odd and $F_{n+1} - 1 < F_n\alpha < F_{n+1}$ if n is even. Therefore, $\lfloor F_n\alpha \rfloor = F_{n+1}$ if n is odd and $\lfloor F_n\alpha \rfloor = F_{n+1} - 1$ if n is even. We consider two cases.

Case 1. n is odd. Then, $\lfloor F_n\alpha \rfloor - 2(F_n - F_m) = 2F_m - F_{n-2}$ and (3.10) becomes

$$\max\{F_m, 2F_m - F_{n-2}\},$$

which is equal to $2F_m - F_{n-2}$ if $2F_m - F_{n-2} \geq F_m$, otherwise, it is F_m. Recall that $3 \leq m \leq n$. Thus, we have $2F_m - F_{n-2} \geq F_m$ if and only if $m \geq n-2$. Therefore,

$$\sum_{0 \leq k < F_m} \left\lfloor \alpha + \frac{k}{F_n} \right\rfloor = \begin{cases} 2F_m - F_{n-2}, & \text{if } m \geq n-2 \text{ and } n \text{ is odd;} \\ F_m, & \text{if } m < n-2 \text{ and } n \text{ is odd.} \end{cases}$$

Case 2. n is even. Similar to Case 1, (3.10) becomes $\max\{F_m, 2F_m - F_{n-2} - 1\}$, and $2F_m - F_{n-2} - 1 \geq F_m$ if and only if $m \geq n-2$. The result follows. □

Recall that a set of positive integers $a_0, a_1, \ldots, a_{n-1}$ is called a complete residue system modulo n if for each $i \in \{0, 1, 2, \ldots, n-1\}$, there exists a unique $j \in \{0, 1, \ldots, n-1\}$ such that $a_j \equiv i \pmod{n}$. Another generalization of Hermite's identity is as follows.

Theorem 3.6. *Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Assume that the set $\{a_0, a_1, \ldots, a_{n-1}\}$ is a complete residue system modulo n. Then,*

$$\sum_{0 \leq k \leq n-1} \left\lfloor x + \frac{a_k}{n} \right\rfloor = \lfloor nx \rfloor - \frac{n-1}{2} + \frac{1}{n}\sum_{k=0}^{n-1} a_k.$$

In particular, if $\{a_0, a_1, \ldots, a_{n-1}\} = \{0, 1, \ldots, n-1\}$, then this is the same as Hermite's identity.

Proof. We can assume, without loss of generality, that for each $k = 0, 1, \ldots, n-1$, we have $a_k \equiv k \pmod{n}$. Then, $(a_k - k)/n$ is an integer and we obtain, by Lemma 2.1(iii), that

$$\left\lfloor x + \frac{a_k}{n} \right\rfloor = \left\lfloor x + \frac{a_k - k}{n} + \frac{k}{n} \right\rfloor = \frac{a_k - k}{n} + \left\lfloor x + \frac{k}{n} \right\rfloor.$$

From this and Theorem 3.1, we see that the sum in question is equal to

$$\frac{1}{n}\left(\sum_{0 \leq k \leq n-1} (a_k - k)\right) + \lfloor nx \rfloor = \lfloor nx \rfloor - \frac{n-1}{2} + \frac{1}{n}\sum_{k=0}^{n-1} a_k.$$ □

Corollary 3.7. *Let $x \in \mathbb{R}$, $a \in \mathbb{Z}$, $n \in \mathbb{N}$, and $(a, n) = 1$. Then,*

$$\sum_{k=0}^{n-1} \left\lfloor x + \frac{ka}{n} \right\rfloor = \frac{(a-1)(n-1)}{2} + \lfloor nx \rfloor.$$

In particular, if $a = 1$, then this is the same as Hermite's identity.

Proof. Since $(a, n) = 1$, the set $\{ka \mid k = 0, 1, \ldots, n-1\}$ is a complete residue system modulo n. So, we can apply Theorem 3.6 to obtain the result. □

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