REPDIGITS IN EULER FUNCTIONS OF PELL NUMBERS

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Abstract. A natural number n is called a repdigit if all its digits are the same. In this paper, we prove that the Euler totient function of no Pell number is a repdigit with at least two digits.

1. Introduction

The Pell sequence $\{P_n\}_{n\geq 0}$ and the associated Pell sequence $\{Q_n\}_{n\geq 0}$ are defined by the binary recurrences

$$P_{n+1} = 2P_n + P_{n-1},\ Q_{n+1} = 2Q_n + Q_{n-1},$$

with the initial terms $P_0 = 0$, $P_1 = 1$ and $Q_0 = 1$, $Q_1 = 1$, respectively. If $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$, then their Binet forms are $P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $Q_n = \frac{\alpha^n + \beta^n}{2}$ for all $n \geq 0$. The Euler totient function $\phi(n)$ of a positive integer n is the number of positive integers less than or equal to n and relatively prime to n. If n has the canonical decomposition $n = p_1^{a_1} \cdots p_r^{a_r}$, then it is well-known that

$$\phi(n) = p_1^{a_1-1}(p_1 - 1) \cdots p_r^{a_r-1}(p_r - 1).$$

In [17], it was shown that if the Euler function of the nth Pell number P_n or associated Pell number Q_n is a power of 2, then $n \leq 8$. In 2014, Damir et al. [6] proved that if $\{u_n\}_{n\geq 0}$ is the Lucas sequence defined by $u_0 = 0$, $u_1 = 1$ and $u_{n+2} = ru_{n+1} + su_n$ for all $n \geq 0$ with $s \in \{1, -1\}$, then there are finitely many n such that $\phi(|u_n|)$ is a power of 2.

A positive integer is called a repdigit if it has only one distinct digit in its decimal expansion. Thus, the repdigits are of the form $d(10^m - 1)/9$ for some $m \geq 1$ and $1 \leq d \leq 9$. In [7], it was shown that there is no repdigit Pell or Pell-Lucas number larger than 10.

The study of repdigits in Euler functions of specified number sequences has attracted several number theorists. In 2002, Luca ([12], p. 134) proved that under some technical assumptions, there exist only finitely many positive integer solution (m, n) satisfying the Diophantine equation $\phi(U_n) = V_m$, where $\{U_n\}_{n\geq 0}$ and $\{V_m\}_{m\geq 0}$ are two non-degenerate binary recurrence sequences. Taking $V_m = d \cdot \frac{10^m - 1}{9}$ where $d \in \{1, 2, \ldots, 9\}$, Luca [3, 16] investigated the presence of repdigits associated with the Euler functions of Fibonacci and Lucas numbers. In this paper, we follow the method described in [3, 16] to investigate the presence of repdigits with at least two digits in the Euler functions of Pell numbers.

Throughout this paper, we use p, with or without subscripts, as a prime number and (a, b) as the greatest common divisor of a and b. If b is odd and $(a, b) = 1$, then we also denote the Jacobi symbol of a and b by $\left(\frac{a}{b}\right)$.

2. Preliminaries

To achieve the objective of this paper, we need the following results and definitions. We shall keep referring to this section with or without further reference.

Lemma 2.1. *If m and n are natural numbers, then*

(1) $P_{2n} = 2P_nQ_n$,
(2) $Q_n^2 - 2P_n^2 = (-1)^n$,
(3) $(P_n, Q_n) = 1$,
(4) $P_m|P_n$ *if and only if* $m|n$,
(5) $v_2(P_n) = v_2(n)$ *and* $v_2(Q_n) = 0$, *where* $v_2(n)$ *is the exponent of* 2 *in the canonical decomposition of* n.

For the proof of this lemma, readers are advised to refer to [9].

Lemma 2.2. ([23], Theorem 2, [5], Theorem 1). *The only solutions of the Diophantine equation* $P_n = y^m$ *in positive integers* n, y, *and* m, *with* $m \geq 2$, *are* $(n, y, m) = (1, 1, m)$, $(7, 13, 2)$.

Lemma 2.3. ([19], Theorem 1). *The solutions of the Diophantine equation* $P_mP_n = x^2$ *with* $1 \leq m < n$ *are* $(m, n) = (1, 7)$ *or* $n = 3m$, $3 \nmid m$, m *is odd.*

Lemma 2.4. ([2], Theorem A). *If* n, y, m *are positive integers with* $m \geq 2$, *then the only solution of equation* $Q_n = y^m$ *is* $(n, y) = (1, 1)$.

Lemma 2.5. *If* m *and* n *are positive integers and* p *is an odd prime, then the Diophantine equation* $P_n = 4p^m$ *has only one integer solution:* $n = 4$, $p = 3$, *and* $m = 1$.

Proof. Suppose that $P_n = 4p^m$ where p is a prime and m and n are positive integers. Since $4|P_n$, $n = 4k$ for some k. Hence, $P_n = P_{4k} = 2P_{2k}Q_{2k} = 4p^m$. Since $(P_{2k}, Q_{2k}) = 1$ and Q_n is odd for all $n \geq 0$, it follows that $P_{2k} = 2$ and $Q_{2k} = p^m$. □

Lemma 2.6. ([2], Lemma 2.1, [25], p. 869). *Let* $(u_n)_{n\geq 0}$ *be a Lucas sequence with* $u_0 = 0$ *and* $u_1 = 1$ *and* $\Delta = (\alpha - \beta)^2$ *be its discriminant. If there exists a prime* p *such that* $p|u_n$ *and* $p \nmid \Delta \cdot \prod_{i=1}^{n-1} u_i$, *then* p *is called as primitive prime factor of* u_n *and is congruent to* ± 1 *modulo* n.

Lemma 2.7. ([2], Lemma 2.1). *A primitive prime factor of* P_n *exists if* $n \geq 3$ *and a primitive prime factor of* Q_n *exists if* $n \geq 2$.

Lemma 2.8. ([18], Theorem 4) *There exist a prime factor* p *of* P_n *such that* $p \equiv 1 \pmod 4$ *if* $n \neq 0, 1, 2, 4, 14$.

Lemma 2.9. ([8], Pell and Pell-Lucas numbers). *If the associated Pell number* Q_n *is a prime, then* n *is either a prime or a power of* 2 *and* P_n *is a prime if and only if* n *is prime.*

3. Repdigits in Euler Functions of Pell Numbers

We begin this section by computing the least residues and periods of the Pell sequence $\{P_n\}_{n\geq 0}$ modulo 5 and associated Pell sequence $\{Q_n\}_{n\geq 0}$ modulo 5, 8. These residues will be used in the proof of Theorem 3.1.

The following theorem, which proves the nonexistence of repdigits with at least two digits in the Euler function of Pell numbers, is the main result of this paper.

Theorem 3.1. *The equation*

$$\phi(P_n) = d \cdot \frac{10^m - 1}{9} \tag{3.1}$$

has no solution in positive integers n, m, d *such that* $m \geq 2$ *and* $d \in \{1, 2, \ldots, 9\}$.

TABLE 1. Periods of P_n

	Least Residues	Period
$P_n \pmod 5$	0, 1, 2, 0, 2, 4, 0, 4, 3, 0, 3, 1	12
P_n (mod 1)1	0, 1, 2, 5, 1, 7, 4, 4, 1, 6, 2, 10, 0, 10, 9, 6, 10, 4, 7, 7, 10, 5, 9, 1	24
P_n (mod 4)0	0, 1, 2, 5, 12, 29, 30, 9, 8, 25, 18, 21, 20, 21, 22, 25, 32, 9, 10, 29, 28, 5, 38,1	24
$Q_n \pmod 5$	1, 1, 3, 2, 2, 1, 4, 4, 2, 3, 3, 4	12
$Q_n \pmod 8$	1, 1, 3, 7	4

Proof. For $n \leq 16$, it is easy to see that there is no Pell number P_n such that $\phi(P_n)$ is a repdigit with at least two digits. Assume to the contrary that for some $n > 16$, $\phi(P_n)$ is a repdigit. That is

$$\phi(P_n) = d \cdot \frac{10^m - 1}{9}$$

for some $d \in \{1, 2, \dots, 9\}$ and for some n. Let $v_2(n)$ be the exponent of 2 in the factorization of a positive integer n. Since $\frac{10^m-1}{9}$ is odd, it follows that

$$v_2(\phi(P_n)) = v_2(d) \leq 3. \tag{3.2}$$

By virtue of Lemma 2.8, there exists a prime factor p_1 of P_n such that $p_1 \equiv 1 \pmod 4$. Clearly, $p_1 - 1 | \phi(P_n)$ and $v_2(d) \geq 2$, which implies that there exists another odd prime factor p_2 of P_n such that $p_2 \equiv 3 \pmod 4$ or p_1 is the only odd prime factor of P_n.

First, assume P_n has two distinct prime factors p_1 and p_2 such that $p_1 \equiv 1 \pmod 4$ and $p_2 \equiv 3 \pmod 4$. If n is odd, reducing relation (2) in Lemma 2.1 modulo p_2, we get $Q_n^2 \equiv -1 \pmod{p_2}$, which implies that -1 is a quadratic residue modulo p_2. But, this is possible only when $p_2 \equiv 1 \pmod 4$, which is a contradiction to $p_2 \equiv 3 \pmod 4$. If n is even, then P_n is even. Let $P_n = 2^a \cdot p_1^b \cdot p_2^c$. If $a > 1$, then $2(p_1 - 1)(p_2 - 1) | \phi(P_n)$, which implies $v_2(\phi(P_n)) \geq 4$. This contradicts (3.2). Hence, $a = 1$ and consequently $P_n = 2 \cdot p_1^b \cdot p_2^c$. Let $n = 2n_1$. If n_1 is even, then $4|n$. Thus, $4|P_n$, so $a \geq 2$. Therefore, n_1 must be odd. Since $P_{2n_1} = 2P_{n_1}Q_{n_1}$ and $(P_{n_1}, Q_{n_1}) = 1$, it follows that $P_{n_1} = p_1^b$ and $Q_{n_1} = p_2^c$.

Since $n_1 > 8$, it follows, from Lemma 2.2 and 2.4, that $P_{n_1} = p_1$, $Q_{n_1} = p_2$ and consequently $P_n = 2p_1p_2$. This implies that $v_2(d) \geq 3$. Hence, the only possible value of d is 8. Since P_{n_1} and Q_{n_1} are primes, it follows, from Lemma 2.9, that n_1 is a prime. Further, reducing $Q_{n_1}^2 - 2P_{n_1}^2 = -1$ modulo p_2, we get $(\frac{2}{p_2}) = 1$, which with $p_2 \equiv 3 \pmod 4$ gives $p_2 \equiv 7 \pmod 8$. Since the period of $\{Q_m\}_{m \geq 0}$ modulo 8 is 4 (see Table 1) and $Q_{n_1} = p_2 \equiv 7 \pmod 8$, it follows that $n_1 \equiv 3 \pmod 4$. Thus, n_1 is of the form $12k + 3$, $12k + 7$, or $12k + 11$. Furthermore, in view of Table 1, the period of both $\{P_m\}_{m \geq 0}$ and $\{Q_m\}_{m \geq 0}$ modulo 5 is 12.

If $n_1 = 12k + 3$, then $p_1 = P_{n_1} \equiv 0 \pmod 5$, which implies that $n_1 = 3$. This contradicts our assumption that $n_1 > 8$.

If $n_1 = 12k + 7$, then $p_1 = P_{n_1} \equiv 4 \pmod 5$ and $p_2 = Q_{n_1} \equiv 4 \pmod 5$ and therefore, $\phi(P_n) = \phi(2P_{n_1}Q_{n_1}) = (p_1 - 1)(p_2 - 1) \equiv 4 \pmod 5$. Since $d = 8$, it follows that $d(10^m - 1)/9 \equiv 3 \pmod 5$. This is a contradiction to the assumption that $\phi(P_n)$ is a repdigit.

If $n_1 = 12k + 11$, then $p_1 = P_{n_1} \equiv 1 \pmod 5$, which implies that $\phi(P_n) \equiv 0 \pmod 5$, but with $d = 8$, $d(10^m - 1)/9 \equiv 3 \pmod 5$. Therefore, $\phi(P_n) \neq d \cdot \frac{10^m-1}{9}$. Hence, there exists only one odd prime factor p_1 of P_n. If n is even (say $n = 2n_1$), then by Lemma 2.1, $P_n = 2P_{n_1}Q_{n_1} = 2^a p_1^b$ and consequently, $P_{n_1} = 2^{a-1}$ and $Q_{n_1} = p_1^b$. If $P_{n_1} = 2^{a-1}$, then in view of Lemmas 2.2 and 2.4, $n_1 \in \{1, 2\}$, which contradicts the assumption that $n > 16$. If n is odd, then $P_n = p_1^b$. If $b \geq 2$, then by virtue of Lemma 2.2, $n \in \{1, 7\}$, which also contradicts our assumption that $n > 16$. If $b = 1$, then $P_n = p_1$ and therefore, $\phi(P_n) = p_1 - 1 = P_n - 1$ is a multiple of 4. Thus, $d \in \{4, 8\}$.

If $d = 4$, then

$$P_n = 4 \cdot \frac{10^m - 1}{9} + 1 = \frac{4 \cdot 10^m + 5}{9}$$

is divisible by 5. This contradicts $P_n = p_1$. If $d = 8$, then (3.1) can be written as

$$9P_n - 1 = 8 \cdot 10^m = 2^{m+3} 5^m. \tag{3.3}$$

Since $m \geq 1$, $9P_n - 1 \equiv 0 \pmod{40}$ and in view of Table 1, this is possible if $n \equiv 7, 17 \pmod{24}$. But, modulo 11, the Pell sequence has period 24. If $n \equiv 7, 17 \pmod{24}$, then $P_n \equiv 4 \pmod{11}$. Reducing (3.3) modulo 11, we get $35 \equiv 8 \cdot 10^m \pmod{11}$. This results in $3 \equiv \pm 1 \pmod{11}$, which is not true. Hence, $\phi(P_n)$ cannot be a repdigit consisting of at least two digits for any natural number n. □

4. Conclusion

From the proof of Theorem 3.1, we can also conclude that the Euler function of none of the odd indexed balancing number B_n is a repdigit with at least two digits, since $P_{2n} = 2B_n$ [1, 22, 24]. Using similar techniques, one can verify that the Euler function of no Lucas-balancing number is a repdigit consisting of more than one digit. Exploring repdigits in Euler function of associated Pell numbers is also equally interesting. We leave these as open problems for the readers.

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