FIBONACCI FUNDAMENTAL SYSTEM AND GENERALIZED CASSINI IDENTITY

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Abstract. In this paper, we explore properties of the Fibonacci fundamental system. The matrix properties of this fundamental system allow us to establish the generalization of the Cassini identity. Further, combinatorial representation of this identity is given and known properties are recovered.

1. Introduction

In the literature, there are several generalizations of the classical sequence $\{F_n\}_{n\geq 0}$ of Fibonacci numbers defined by $F_{n+1} = F_n + F_{n-1}$, for $n \geq 1$, where $F_0 = 1$ and $F_1 = 0$ or $F_0 = 0$ and $F_1 = 1$. In this study, we are concerned with the generalization defined by the following linear difference equation of order r,

$$F_n = \sum_{i=0}^{r-1} F_{n-i-1} \qquad \text{for} \qquad n \geq r, \tag{1.1}$$

where the initial conditions $F_0, \ldots, F_{r-1}$ are chosen adequately. Among the solutions of equation (1.1) we specify the family of generalized Fibonacci numbers $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n\geq 0},\ 1 \leq s \leq r\}$, defined by

$$F_n^{(s)} = \sum_{i=0}^{r-1} F_{n-i-1}^{(s)} \qquad \text{for} \qquad n \geq r, \tag{1.2}$$

where the initial conditions $F_n^{(s)}$, for $n = 0,\ 1, \ldots,\ r-1$, are given by $F_{s-1}^{(s)} = 1$ and $F_n^{(s)} = 0$ for $0 \leq n \neq s-1 \leq r-1$. In the current literature, the most studied among the sequences of the set $\mathfrak{F} = \{\{F_n^{(s)}\}_{n\geq 0},\ 1 \leq s \leq r\}$ are $\{F_n^{(1)}\}_{n\geq 0}$ and $\{F_n^{(r)}\}_{n\geq 0}$, namely, the sequences of generalized Fibonacci numbers, whose initial conditions are given by $F_0^{(1)} = 1$ and $F_n^{(1)} = 0$ for $1 \leq n \leq r-1$ or $F_{r-1}^{(r)} = 1$ and $F_n^{(r)} = 0$ for $0 \leq n \leq r-2$.

In this paper, we are interested in the fundamental role of the family of generalized Fibonacci numbers of the set $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n\geq 0},\ 1 \leq s \leq r\}$. Principally, we describe explicitly the close connection between the family of generalized Fibonacci numbers $\{F_n^{(s)}\}_{n\geq 0}$, for $2 \leq s \leq r-2$, and the the two fundamental generalized Fibonacci numbers $\{F_n^{(1)}\}_{n\geq 0}$ and $\{F_n^{(r)}\}_{n\geq 0}$. As a consequence, we derive various new identities related to generalized Fibonacci numbers. Moreover, we give a generalization of the Cassini identity and some related combinatorial identities.

The content of this paper is organized as follows. In Section 2, we establish that the set $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n\geq 0},\ 1 \leq s \leq r\}$ is a fundamental set of solutions for the difference equation (1.1). Moreover, the main role of the sequences $\{F_n^{(1)}\}_{n\geq 0}$ and $\{F_n^{(r)}\}_{n\geq 0}$ as important solutions of equation (1.1) is studied. Section 3 is devoted to some identities related to the Fibonacci fundamental system. Some combinatorial identities are established. Section 4 concerns Cassini identity and its generalization. A related combinatorial expression is proposed.

2. Study of the Fibonacci Fundamental System

2.1. **Fibonacci Fundamental System.** For the general Fibonacci difference equation (1.1), we show that the set of generalized Fibonacci numbers sequences, $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n\geq 0},\ 1 \leq s \leq r\}$, represents r copies of solutions of equation (1.1), with mutually different sets of initial values, $F_j^{(s)} = \delta_{s-1,j}$ $(0 \leq j \leq r-1,\ 1 \leq s \leq r)$, where $\delta_{i,j}$ is the Kronecker symbol. These copies can be represented by the following compact form,

$$\begin{cases} F_{n+1}^{(s)} = F_n^{(s)} + \cdots + F_{n-r+1}^{(s)}; & \text{for } n \geq r-1, \\ F_n^{(s)} = \delta_{s-1,n}; & \text{for } 0 \leq n \leq r-1. \end{cases} \tag{2.1}$$

And for a given solution $\{F_n\}_{n\geq 0}$ of equation (1.1), with initial conditions $F_0 = \alpha_0, \ldots,\ F_{r-1} = \alpha_{r-1}$, we can verify easily that $F_n = \sum_{s=1}^{r} \alpha_s F_n^{(s)}$ for every $n \geq 0$. In other words, the set $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$ is a system generator for the vector space (over $\mathbb{R}$) $\mathcal{E}_{\mathbb{K}}^{(r)}$ of solutions of equation (1.1). We will establish that $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n\geq 0}\ ;\ 1 \leq s \leq r\}$ is a fundamental system of solutions for equation (1.1), namely, the sequences of generalized Fibonacci numbers $\{F_n^{(1)}\}_{n\geq 0}, \ldots, \{F_n^{(r)}\}_{n\geq 0}$ are linearly independent. To this end, we consider the Casoratian matrix (see for example [2, 5]) associated with the set $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$ given by

$$\widehat{C}(n) = \begin{pmatrix} F_n^{(1)} & \cdots & F_n^{(j)} & \cdots & F_n^{(r)} \\ \vdots & \vdots & \vdots & & \vdots \\ F_{n+r-1}^{(1)} & \cdots & F_{n+r-1}^{(j)} & \cdots & F_{n+r-1}^{(r)} \end{pmatrix}. \tag{2.2}$$

A direct verification shows that the Casoratian matrix (2.2) can be written in the form

$$\widehat{C}(n) = J \times \mathbb{M}_n \times J, \text{ where } \mathbb{M}_n = \begin{pmatrix} F_{n+r-1}^{(r)} & \cdots & F_{n+r-1}^{(j)} & \cdots & F_{n+r-1}^{(1)} \\ \vdots & \vdots & \vdots & & \vdots \\ F_n^{(r)} & \cdots & F_n^{(j)} & \cdots & F_n^{(1)} \end{pmatrix}, \tag{2.3}$$

where $J = (b_{i,j})_{1\leq i,j\leq r}$ is the antidiagonal unit matrix whose entries are given by $b_{i,j} = 1$ for $i + j = r + 1$ and $b_{i,j} = 0$ otherwise. To begin our result, we recall the following useful lemma.

Lemma 2.1. *(Rachidi et al.,* [1, 3]*) For every $n \geq 0$, we have*

$$\mathbb{M}_n = \mathbb{A}^n,$$

where $\mathbb{A}$ is the classical companion matrix

$$\mathbb{A} = \begin{pmatrix} 1 & 1 & & \cdots & 1 \\ 1 & 0 & & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{2.4}$$

Expression $\mathbb{A}^n$ given in Lemma 2.1 has been improved in [4, 6], when the companion matrix $\mathbb{A}$ (of order $r = 2$) is related to the usual Fibonacci numbers. Combining (2.2) and Lemma 2.1, we derive the following property.

Proposition 2.2. *Consider the set $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$ of sequences of generalized Fibonacci numbers (2.1). Then, the Casoratian matrix $\widehat{C}(n)$ of the $\mathfrak{F}_r$ and the matrix powers $\mathbb{A}^n$ of the companion matrix (2.4) are similar. More precisely, we have*

$$\widehat{C}(n) = J\mathbb{A}^n J = (c_{ij}^{(n)})_{1\leq i,\, j\leq r}, \tag{2.5}$$

where the entries $c_{ij}^{(n)}$ are given by $c_{ij}^{(n)} = F_{n+i-1}^{(j)}$ $(1 \leq i, j \leq r)$ and $J = (b_{i,\,j})_{1\leq i,\, j\leq r}$ is the antidiagonal unit matrix, whose entries are given by $b_{i,\,j} = 1$ for $i + j = r + 1$ and $b_{i,\,j} = 0$ otherwise.

Theorem 2.3. *The Casoratian, $C(n)$, of the set $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$ of sequences of generalized Fibonacci numbers (2.1), is given by*

$$C(n) = det[\widehat{C}(n)] = det[\mathbb{A}^n] = det[\mathbb{A}]^n = (-1)^{n(r-1)} \neq 0. \tag{2.6}$$

Consequently, the set $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$ is a fundamental system of solution for equation (1.1).

The set $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$ is called the *Fibonacci fundamental system*.

For example, in the case $r = 3$, the fundamental system $\mathfrak{F}_3 = \{\{F_n^{(s)}\}_{n\geq 0};\ 1 \leq s \leq 3\}$ is given by

$$\begin{cases} F_{n+1}^{(s)} = F_n^{(s)} + F_{n-1}^{(s)} + F_{n-2}^{(s)}, & \text{for } n \geq 2, \\ F_n^{(s)} = \delta_{s-1,n} & \text{for } 0 \leq n \leq 2, \end{cases} \tag{2.7}$$

with values described in Table 1.

n	0	1	2	3	4	5	6	7	8	9	10	11	...
$F_n^{(1)}$	1	0	0	1	1	2	4	7	13	24	44	81	...
$F_n^{(2)}$	0	1	0	1	2	3	6	11	20	37	68	125	...
$F_n^{(3)}$	0	0	1	1	2	4	7	13	24	44	81	149	...

Table 1: List of generalized Fibonacci number of order $r = 3$.

Then, the Casoratian matrix associated with the set $\mathfrak{F}_3 = \{\{F_n^{(s)}\}_{n\geq 0};\ 1 \leq s \leq 3\}$ is

$$\widehat{C}(n) = \begin{pmatrix} F_n^{(1)} & F_n^{(2)} & F_n^{(3)} \\ F_{n+1}^{(1)} & F_{n+1}^{(2)} & F_{n+1}^{(3)} \\ F_{n+2}^{(1)} & F_{n+2}^{(2)} & F_{n+2}^{(3)} \end{pmatrix}, \tag{2.8}$$

or

$$\widehat{C}(n) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \tag{2.9}$$

with $C(n) = det[\widehat{C}(n)] = det\left[\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right]^n = 1.$

By combining Lemma 2.1 and Proposition 2.2, we show that $\widehat{C}(n+m) = J\mathbb{A}^n\mathbb{A}^m J = J\mathbb{A}^n J \cdot J\mathbb{A}^m J$, because the antidiagonal unit matrix J satisfies $J \cdot J = diag(1, \ldots, 1)$. Hence, we obtain the following property of the Casoratian matrix.

Corollary 2.4. *Let $\widehat{C}(n)$ be the Casoratian matrix of the Fibonacci fundamental system* (2.1)*. Then, for every n and m, we have the identity,*

$$\widehat{C}(n+m) = \widehat{C}(n).\widehat{C}(m). \tag{2.10}$$

Remark 2.5. *We can establish that the set $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$ of sequences of generalized Fibonacci numbers is a fundamental system of solution for equation (1.1), using some techniques of linear algebra. However, the usual method based on the Casoratian matrix and the Casoratian is useful in the next sections.*

2.2. **Properties of the Fibonacci fundamental system.** In this subsection, we are interested in the study of the Fibonacci fundamental system. Some results are developed and some classical results are obtained. To reach our goal, we start with the following elementary lemma.

Lemma 2.6. *Let $\mathcal{E}_{\mathbb{K}}^{(r)}$ be the vector space (over $\mathbb{R}$) of solutions of equation (1.1). Consider two sequences $\{F_n^{[C_1]}\}_{n\geq 0}$ and $\{F_n^{[C_2]}\}_{n\geq 0}$ of $\mathcal{E}_{\mathbb{K}}^{(r)}$, whose initial conditions are $C_1 = (\alpha_0, \alpha_1, \ldots, \alpha_{r-1})$ and $C_2 = (\beta_0, \beta_1, \ldots, \beta_{r-1})$, respectively. Suppose there exist n_0, m_0, and N in $\mathbb{N}$ such that*

$$F_{j+n_0}^{[C_1]} = F_{j+m_0}^{[C_2]} \textit{for } N \leq j \leq N+r-1. \tag{2.11}$$

Then, we have $F_{n+n_0}^{[C_1]} = F_{n+m_0}^{[C_2]}$ for every $n \geq N$.

Proof. Suppose there exist n_0, m_0, and N in $\mathbb{N}$ such that $F_{n+n_0}^{[C_1]} = F_{n+m_0}^{[C_2]}$ for $N \leq n \leq N+r-1$. Then, for $n = N+r$, we have

$$\begin{aligned} F_{N+r+n_0}^{[C_1]} &= F_{N+r-1+n_0}^{[C_1]} + F_{N+r-2+n_0}^{[C_1]} + \cdots + F_{N+n_0}^{[C_1]} \\ &= F_{N+r-1+m_0}^{[C_2]} + F_{N+r-2+m_0}^{[C_2]} + \cdots + F_{N+m_0}^{[C_2]} \\ &= F_{N+r+m_0}^{[C_2]}. \end{aligned} \tag{2.12}$$

Therefore, we show that $F_{N+r+n_0}^{[C_1]} = F_{N+r+m_0}^{[C_2]}$. And, an induction process in n allows us to obtain $F_{N+n+n_0}^{[C_1]} = F_{N+n+m_0}^{[C_2]}$ for every $n \geq N$. □

Our study of the Fibonacci fundamental systems $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$ is based two main questions: "*What is the closed connection between the components $\{F_n^{(s)}\}_{n\geq 0}$ of the Fibonacci fundamental systems $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$? What is the role of the two basic sequences of generalized Fibonacci numbers $\{F_n^{(1)}\}_{n\geq 0}$ and $\{F_n^{(r)}\}_{n\geq 0}$?*"

Consider two basic sequences $\{F_n^{(1)}\}_{n\geq 0}$ and $\{F_n^{(r)}\}_{n\geq 0}$. For $n = 1, \ldots, r-1, r$ we have

$$F_1^{(1)} = \cdots = F_{r-1}^{(1)} = 0, F_r^{(1)} = 1 \text{ and } F_0^{(r)} = F_1^{(r)} = \cdots = F_{r-2}^{(r)} = 0, F_{r-1}^{(r)} = 1. \tag{2.13}$$

Hence, we observe that $F_{j+1}^{(1)} = F_j^{(r)} = 0$ for $j = 0, \ldots, r-2$ and $F_r^{(1)} = F_{r-1}^{(r)} = 1$. Hence, applying Lemma 2.6 allows us to obtain $F_{n+1}^{(1)} = F_n^{(r)}$ for every $n \geq 0$. Thus, we have the following proposition.

Proposition 2.7. *Let $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$ be the Fibonacci fundamental system. Then, we have*

$$F_{n+1}^{(1)} = F_n^{(r)} \text{ for every } n \geq 0 \text{ or equivalently } F_n^{(1)} = F_{n-1}^{(r)} \text{ for every } n \geq 1. \quad (2.14)$$

For $r = 4$ and $4 \leq n \leq 11$, Proposition 2.7 can be verified in Table 1. Now, what about the closed relation between the two sequences $\{F_n^{(1)}\}_{n\geq 0}$ and $\{F_n^{(r)}\}_{n\geq 0}$ with the other sequences $\{F_n^{(s)}\}_{n\geq 0}$ for $2 \leq s \leq r-1$? For reason of clarity, consider the case $r = 4$. A simple computation shows that the first terms of the $F_n^{(j)}$ for $1 \leq j \leq 4$ are given in Table 2.

n	0	1	2	3	4	5	6	7	8	9	10	11	...
$F_n^{(1)}$	1	0	0	0	1	1	2	4	8	15	29	56	...
$F_n^{(2)}$	0	1	0	0	1	2	3	6	12	23	44	85	...
$F_n^{(3)}$	0	0	1	0	1	2	4	7	14	27	52	100	...
$F_n^{(4)}$	0	0	0	1	1	2	4	8	15	29	56	108	...

Table 2: List of generalized Fibonacci number of order $r = 4$.

Following Table 2, we show that for $4 \leq n \leq 11$,

$$F_n^{(2)} = F_n^{(1)} + F_{n-1}^{(1)} = F_{n-1}^{(4)} + F_{n-2}^{(4)} \text{ and } F_n^{(3)} = F_n^{(1)} + F_{n-1}^{(1)} + F_{n-2}^{(1)} = F_{n-1}^{(4)} + F_{n-2}^{(4)} + F_{n-3}^{(4)}. \quad (2.15)$$

This result suggests that for $r = 4$ and $2 \leq j \leq 3$, the general term $F_n^{(j)}$ is expressed in terms of $F_n^{(1)}$ or $F_n^{(4)}$ as $F_n^{(j)} = F_n^{(1)} + \cdots + F_{n-j+1}^{(1)}$ for $n \geq j-1$ or equivalently $F_n^{(j)} = F_{n-1}^{(4)} + \cdots + F_{n-j}^{(4)}$ for $n \geq j$. We can establish that the conjecture is verified for $r \geq 3$ and, for fixed r, $2 \leq j \leq r-1$. We have the following result.

Proposition 2.8. *Let $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$ be the Fibonacci fundamental system. Then, for every j $(2 \leq j \leq r-1)$, the general term $F_n^{(j)}$ of sequence of generalized Fibonacci numbers $\{F_n^{(j)}\}_{n\geq 0}$ is expressed in terms of the sequence of generalized Fibonacci numbers $F_n^{(1)}$ or $F_n^{(r)}$ as follows,*

$$F_n^{(j)} = F_n^{(1)} + \cdots + F_{n-j+1}^{(1)} \text{ for } n \geq j-1 \text{ or } F_n^{(j)} = F_{n-1}^{(r)} + \cdots + F_{n-j}^{(r)} \text{ for } n \geq j. \quad (2.16)$$

Proof. We proceed by induction. To show that expression (2.16) is valid for $j = 2$, we set $w_n^{(2)} = F_n^{(1)} + F_{n-1}^{(1)}$ for $n \geq 1$. It is clear that the sequence $\{w_n^{(2)}\}_{n\geq 1}$ satisfies (1.1) with initial conditions

$$w_1^{(2)} = 1, w_s^{(2)} = 0 \text{ for } 2 \leq s \leq r-1 \text{ and } w_r^{(2)} = 1. \quad (2.17)$$

For $\{F_n^{(2)}\}_{n\geq 0}$, we have

$$F_1^{(2)} = 1, F_s^{(2)} = 0 \text{ for } 0 \leq s \leq r-1 \text{ and } F_r^{(2)} = 1. \quad (2.18)$$

Hence, we have $F_1^{(2)} = w_1^{(2)} = 1$, $F_s^{(2)} = w_s^{(2)} = 0$ for $0 \leq s \leq r-1$, and $F_r^{(2)} = w_r^{(2)} = 1$. By Lemma 2.6, we derive that $F_n^{(2)} = w_n^{(2)}$ for every $n \geq 1$. Thus, we obtain

$$F_n^{(2)} = F_n^{(1)} + F_{n-1}^{(1)} \quad \text{for every } n \geq 1. \quad (2.19)$$

For $3 \leq j \leq r-1$, suppose that $F_n^{(j)} = F_n^{(1)} + \cdots + F_{n-j+1}^{(1)}$ for every $n \geq j-1$. Consider the sequence $\{w_n^{(j+1)}\}_{n\geq 1}$ defined by $w_n^{(j+1)} = F_n^{(1)} + F_{n-1}^{(j)}$ for every $n \geq j$, since $F_{n-1}^{(j)}$ is defined for $n-1 \geq j-1$. For $F_{n-1}^{(j)}$, the first r terms are $F_{n-1}^{(j)} = 0$ for $1 \leq n \neq j \leq r-1$, $F_j^{(j)} = 1$, and $F_r^{(j)} = 1$. Since $F_0^{(1)} = 1$ and $F_n^{(1)} = 0$ for $n = 1, \ldots, r-1$, by summing $F_n^{(1)} + F_{n-1}^{(j)}$ $(1 \le n \le r)$, we derive that

$$w_n^{(j+1)} = 0 \text{ for } \; 1 \le n \ne j \le r-1 \text{ and } w_j^{(j+1)} = 1. \tag{2.20}$$

Comparing expression (2.20) with the sequence $\{F_n^{(j+1)}\}_{n\ge 0}$ shows that

$$w_n^{(j+1)} = F_n^{(j+1)} = 0 \text{ for } 1 \le n \ne j \le r-1,\; w_j^{(j+1)} = F_j^{(j+1)} = 1, \text{ and } w_r^{(j+1)} = F_r^{(j+1)} = 1. \tag{2.21}$$

Since the sequence $\{w_n^{(j+1)}\}_{n\ge 1}$ satisfies expression (1.1), Lemma 2.6 allows us to get $F_n^{(j+1)} = w_n^{(j+1)} = F_n^{(1)} + w_n^{(j)}$. Therefore, we derive that $F_n^{(j+1)} = F_n^{(1)} + \cdots + F_{n-j+1}^{(1)}$ for every $n \ge j$. □

Propositions 2.7 and 2.8 explain the closed relationship between the elements of the Fibonacci fundamental system of sequences of the generalized Fibonacci numbers (2.1) of order $r \ge 3$, especially, the important role of the two basic sequences $\{F_n^{(1)}\}_{n\ge 0}$ and $\{F_n^{(r)}\}_{n\ge 0}$. Therefore, for reason of simplicity, we adopt the notation

$$F_{n-1} = F_n^{(1)} = F_{n-1}^{(r)} \text{ for every } n \ge 1 \text{ or equivalently } F_n = F_{n+1}^{(1)} = F_n^{(r)} \text{ for every } n \ge 0. \tag{2.22}$$

3. Some Identities and Their Combinatorial Aspect

3.1. **Some Identities Related to the Fibonacci Fundamental System.** As a consequence of Propositions 2.2, 2.7, and 2.8, we exhibit some identities satisfied by the elements of the Fibonacci fundamental system. Lemma 2.1 shows that the entries $a_{i,j}^{(n)}$ of the matrix powers $\mathbb{A}^n$ are given by the following compact formula.

$$a_{i,j}^{(n)} = F_{n+r-i}^{(r-j+1)}. \tag{3.1}$$

Consider the two powers matrices $\mathbb{A}^m = (a_{ij}^{(m)})_{1\le i,\, j\le r}$ and $\mathbb{A}^n = (a_{ij}^{(n)})_{1\le i,\, j\le r}$, for $m \ge 0$ and $n \ge 0$ in $\mathbb{N}$. Since $\mathbb{A}^{m+n} = \mathbb{A}^m\mathbb{A}^n = \mathbb{A}^n\mathbb{A}^m = (a_{ij}^{(m+n)})_{1\le i,\, j\le r}$, we derive

$$a_{ij}^{(m+n)} = \sum_{k=1}^{r} a_{ik}^{(m)} a_{kj}^{(n)} = \sum_{k=1}^{r} a_{ik}^{(n)} a_{kj}^{(m)}. \tag{3.2}$$

Application of this identity and expression (3.1) permits us to obtain the following formula related to the sequences of the Fibonacci fundamental system,

$$F_{m+n+p}^{(q)} = \sum_{d=1}^{r} F_{m+p}^{(d)} F_{n+d-1}^{(q)} = \sum_{d=1}^{r} F_{n+p}^{(d)} F_{m+d-1}^{(q)} \tag{3.3}$$

for any integers $m, n \ge 0$ and every p, q $(1 \le p,\, q \le r)$. According to Proposition 2.8, we know that $F_n^{(j)} = F_{n-1}^{(r)} + \cdots + F_{n-j}^{(r)} = F_{n-1} + \cdots + F_{n-j}$ for $2 \le j \le r-1$ and $n \ge j$. Therefore, for $1 \le q \le r-1$, we have

$$F_{m+n+p}^{(q)} = \sum_{d=1}^{r} \left[\sum_{i=1}^{d} F_{m+p-i} \right] \left[\sum_{j=1}^{q} F_{n+d-j-1} \right]. \tag{3.4}$$

For $q = r$, we obtain

$$F^{(r)}_{m+n+p} = \sum_{d=1}^{r} F^{(d)}_{m+p} F^{(r)}_{n+d-1} = \sum_{d=1}^{r} F^{(d)}_{n+p} F^{(r)}_{m+d-1}. \tag{3.5}$$

In summary, we have the following result.

Theorem 3.1. *Let $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$ be the Fibonacci fundamental system. Then, for every n and m ($n \geq 0$ and $m \geq 0$) and q ($1 \leq q \leq r$), we have the following identities.*

$$F_{m+n} = \sum_{d=1}^{r}\sum_{j=1}^{d} F_{m-j} F_{n+d-1} \tag{3.6}$$

$$\text{and} \sum_{k=1}^{q} F_{n+n-k} = \sum_{d=1}^{r} \sum_{1\leq i\leq d,\ 1\leq j\leq q} F_{n-i} F_{n+d-j-1}. \tag{3.7}$$

To illustrate the content of Theorem 3.1, we consider cases $r = 2$ and $r = 3$. A direct computation using (3.6) allows us to obtain the corollary.

Corollary 3.2. *The usual Fibonacci numbers $\{F_n\}_{n\geq 0}$ ($r = 2$) verify the identity,*

$$F_{m+s} = F_{m-1}F_s + F_m F_{s+1}, \tag{3.8}$$

for every $n \geq 2$ and $s \geq 0$. The generalized Fibonacci numbers $\{F_n\}_{n\geq 0}$ of order $r = 3$ satisfy the identity

$$F_{m+s} = F_m F_{s+2} + (F_{m-1} + F_{m-2})F_{s+1} + F_{m-1}F_s, \tag{3.9}$$

for every $n \geq 2$ and $s \geq 0$.

Formulas (3.8) and (3.9) show that the computation of F_{m+s} can be derived from the knowledge of F_m, F_{m-1}, F_{s+1}, and F_s. Similarly, formula (3.8) permits us to compute F_{m+s} from the knowledge of F_m, F_{m-1}, F_{m-2}, F_{s+2}, F_{s+1}, and F_s.

3.2. **Combinatorial aspect of the Fibonacci fundamental system.** Several methods were used to establish combinatorial expressions for generalized Fibonacci sequences (see [7, 8, 9]). Let $\{u_n\}_{n\geq 0}$ be the sequence defined by

$$u_n = \rho(n+1, r) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-r+1} \frac{(k_0 + \cdots + k_{r-1})!}{k_0!k_1!\cdots k_{r-1}!} \text{ for } n \geq r, \tag{3.10}$$

with $u_{r-1} = \rho(r, r) = 1$ and $u_r = \rho(n, r) = 0$ for $0 \leq n \leq r$. Using the identity

$$\frac{(k_0 + \cdots + k_{r-1})!}{k_0!k_1!\cdots k_{r-1}!} = \sum_{i=0}^{r-1} \frac{(k_0 + \cdots + k_{r-1} - 1)!}{k_0!k_1!\cdots k_{i-1}!(k_i - 1)!k_{i+1}\cdots k_{r-1}!}, \tag{3.11}$$

we derive that

$$u_n = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n} \sum_{i=0}^{r-1} \frac{(k_0 + \cdots + k_{r-1} - 1)!}{k_0!\ldots k_{i-1}!(k_i - 1)!k_{i+1}\cdots k_{r-1}!} = \sum_{i=0}^{r-1} u_{n-i} \tag{3.12}$$

for every $n \geq r$.

Proposition 3.3. *(Combinatoric expression of generalized Fibonacci numbers) Consider the generalized Fibonacci numbers* $\{F_n\}_{n\geq 0}$*, where* $F_n = F_n^{(r)}$*. Then, we have the combinatoric expression,*

$$F_n = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-r+1} \frac{(k_0+\cdots+k_{r-1})!}{k_0!k_1!\cdots k_{r-1}!} \text{ for } n \geq r, \tag{3.13}$$

where $F_{r-1} = F_{r-1}^{(r)} = 1$ *and* $F_n = F_n^{(r)} = 0$ *for* $0 \leq n \leq r-2$*.*

For example, when $r = 2$, the sequence $\{F_n^{(2)}\}_{n\geq 0}$ represents the well-known sequence of Fibonacci numbers of order 2, whose initial conditions are $F_n^{(2)} = \delta_{1,n}$ for $n = 0, 1$. Therefore, we recover the known combinatorial expression given by

$$F_n^{(2)} = \rho(n+1,2) = \sum_{k_0+2k_1=n-1} \frac{(k_0+k_1)!}{k_0!k_1!} \text{ for } n \geq 2, \tag{3.14}$$

with $F_0^{(2)} = 0$ and $F_1^{(2)} = 1$. Expression (3.14) can also take the following form,

$$F_n^{(2)} = \sum_{k=0}^{n-1} \sum_{p=0}^{\left[\frac{n-k-1}{2}\right]} \binom{k+p}{k \;\; p} \text{ for } n \geq 2, \tag{3.15}$$

where $\binom{k+p}{k \;\; p} = \frac{(k+p)!}{k!\,p!}$ and $[x]$ means the integer part of x.

If we let $r = 3$, the sequence $\{F_n^{(3)}\}_{n\geq 0}$ represents the generalized Fibonacci numbers of order 3 with the initial conditions $F_n^{(3)} = \delta_{2,n}$ for $n = 0,\ 1,\ 2$. Then, its combinatorial expression is given by,

$$F_n^{(3)} = \rho(n+1,3) = \sum_{k_0+2k_1+3k_2=n-2} \frac{(k_0+k_1+k_2)!}{k_0!k_1!k_2!} \text{ for } n \geq 3, \tag{3.16}$$

with $F_0^{(3)} = F_1^{(3)} = 0$ and $F_2^{(3)} = 1$ (see more values in Table 1). Expression (3.16) can also take a form similar to (3.15).

More generally, a direct application of Propositions 2.7 and 2.8 leads to the combinatorial formulation of the elements of the Fibonacci fundamental system.

Proposition 3.4. *Let* $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$ *be the Fibonacci fundamental system. The combinatorial expression of each element* $\{F_n^{(s)}\}_{n\geq 0}$*, where* $1 \leq s \leq r$*, is given by*

$$F_n^{(r)} \;=\; F_n = \rho(n+1,r) \textit{ for } n \geq r+1 \tag{3.17}$$

and

$$F_n^{(s)} \;=\; \sum_{j=1}^{s} \rho(n+s-j,r) \quad \textit{for } 2 \leq s \leq r, \tag{3.18}$$

with $n \geq r+s$*, where the* $\rho(n,r)$ *are given by* (3.10)*.*

Proof. Since $F_n^{(1)} = F_{n-1}^{(r)} = F_{n-1}$, we obtain $F_n^{(1)} = F_{n-1} = \rho(n,r)$ for every $n \geq r+1$. For $2 \leq j \leq r-1$, formulas (2.16) and (3.10) give expression (3.17), namely,

$$F_n^{(s)} = F_{n-1} + \cdots + F_{n-s} = \sum_{j=1}^{s} \rho(n-j+1,r). \tag{3.19}$$

□

A direct application of Theorem 3.1 and Proposition 3.4 establishes some combinatorial identities involving the expressions of $\rho(n,j)$ and $\rho(n,r)$. More precisely, we have the combinatorial identities given in the following corollary.

Corollary 3.5. *The combinatorial expressions of the generalized Fibonacci numbers identities* (3.6) *and* (3.7) *are given by*

$$\rho(m+s+1,r) = \sum_{d=1}^{r}\sum_{j=1}^{d} \rho(m-j+1,r)\rho(s+d,r) \tag{3.20}$$

and

$$\sum_{k=1}^{q} \rho(n+s-k+1,r) = \sum_{d=1}^{r}\sum_{1\leq i\leq d,\ 1\leq j\leq q} \rho(n-i+1,r)\rho(s+d-j,r). \tag{3.21}$$

It seems that the combinatorial expressions (3.20) and (3.21) are not in the current literature. In the particular case of $r=2$, the formulas of Corollary 3.5 show that the combinatorial identity (3.20) takes the form,

$$\rho(m+s+1,r) = \rho(m,r)\rho(s+1,r) + \rho(m+1,r)\rho(s,r) \tag{3.22}$$

for every $n \geq 2$ and $s \geq 0$. And for $r=3$, we have the identity

$$\rho(m+s+1,r) = \rho(m+1,r)\rho(s+2,r) + (\rho(m,r) + \rho(m-1,r)\rho(s,r) + \rho(m,r)\rho(s+1,r) \tag{3.23}$$

for every $n \geq 2$ and $s \geq 0$.

4. Generalized Cassini Identity of the Fibonacci Fundamental System

4.1. **Generalized Cassini Identity.** Proposition 2.2 gives the Casoratian matrix $\widehat{C}(n)$ associated with the Fibonacci fundamental system

$$\widehat{C}(n) = J \times \mathbb{A}^n \times J,$$

where J is the antidiagonal unit matrix given in Proposition 2.2 and $\mathbb{A}$ is the companion matrix (2.4) . Therefore, the Casoratian of the Fibonacci fundamental system $\{\{F_n^{(j)}\}_{n\geq 0},\ 1 \leq j \leq r\}$ is given by

$$C(n) = det[\widehat{C}(n)] = det[\mathbb{A}]^n = (-1)^{n(r-1)}. \tag{4.1}$$

For $r=2$, Proposition 2.7 shows that $F_{n+1}^{(1)} = F_n^{(2)} = F_n$, where $\{F_n\}_{n\geq 0}$ is the sequence of usual Fibonacci numbers. Thus, the Casoratian (4.1) is given by

$$C(n) = det[\mathbb{A}^n] = \begin{vmatrix} F_{n+1}^{(2)} & F_{n+1}^{(1)} \\ F_n^{(2)} & F_n^{(1)} \end{vmatrix} = \begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = F_{n+1}F_{n-1} - F_n^2 = (-1)^n. \tag{4.2}$$

Therefore, Cassini identity is given by the Casoratian of the Fibonacci fundamental system related to the sequence of Fibonacci numbers. It seems natural to assert that the generalized Cassini identity is the Casoratian of the Fibonacci fundamental systems. However, in expression (4.1), it is clear that the Casoratian depends on all the elements of the Fibonacci fundamental system. Because of this, it cannot give an adequate generalization of Cassini identity. Expression (4.1) of the Casoratian must be expressed only with the aid of the generalized Fibonacci numbers. To state our generalization, some useful algebraic properties of the determinant of matrix will be used. We consider the family of Fibonacci column vectors,

$$\overrightarrow{F}(j,n) = (F_n^{(j)},\ F_{n+1}^{(j)}, \dots,\ F_{n+r-1}^{(j)})^t \quad \text{for } n \geq r-1, \tag{4.3}$$

where $1 \leq j \leq r$. It is easy to show that the Fibonacci Casoratian matrix is

$$\widehat{C}(n) = [\overrightarrow{F}(1,n), \dots, \overrightarrow{F}(j,n), \dots, \overrightarrow{F}(r,n)], \tag{4.4}$$

where $\overrightarrow{F}(j,n)$ $(1 \leq j \leq r)$ represents the vector columns. Now, the results of Proposition 2.8 imply that the the vectors column $\overrightarrow{F}(j,n)$ $(1 \leq j \leq r)$ satisfy the following recursive relation:

$$\begin{aligned} \overrightarrow{F}(r,n) &= \overrightarrow{F}(1,n+1), \\ \overrightarrow{F}(j,n) &= \overrightarrow{F}(1,n) + \cdots + \overrightarrow{F}(1,n-j+1) \text{ for } 2 \leq j \leq r, \end{aligned} \tag{4.5}$$

where $n \geq j+1$. For $2 \leq j \leq r-1$, the former expression can be written in the form

$$\overrightarrow{F}(j,n) = \sum_{s=0}^{j-1} \overrightarrow{F}(1,n-s) = \overrightarrow{F}(j-1,n) + \overrightarrow{F}(1,n-j+1) \tag{4.6}$$

for $n \geq j-1$. For $j = r-1$, expression (4.6) and the determinant identity $det(\overrightarrow{x}_1, \dots, \overrightarrow{x}_{s-1}, \overrightarrow{y}, \overrightarrow{y}, \overrightarrow{x}_{s+2}, \dots, \overrightarrow{x}_r) = 0$ show that the Casoratian of the Fibonacci fundamental system can be reduced to the form

$$C(n) = det([\overrightarrow{F}(1,n), \overrightarrow{F}(2,n), \dots,, \overrightarrow{F}(r-2,n), \overrightarrow{F}(1,n-r+2), \overrightarrow{F}(r,n)]) \tag{4.7}$$

for every $n \geq r$. By iteration of the preceding process to the other vectors column $\overrightarrow{F}(j,n)$ $(2 \leq j \leq r-2)$ and taking into account that $F_{n+1}^{(1)} = F_n^{(r)}$, we establish that

$$C(n) = det([\overrightarrow{F}(1,n), \dots, \overrightarrow{F}(1,n-j), \dots, \overrightarrow{F}(1,n-r+2), \overrightarrow{F}(1,n+r-1)]). \tag{4.8}$$

In (4.8), the vectors column $\overrightarrow{F}(1,n-j)$ $(0 \leq j \leq r-1)$ must be in descending order from left to right in the expression of the determinant. Specifically, the Cassini identity is obtained from the Casoratian $C(n)$ as

$$det([\overrightarrow{F}(1,n+1), \overrightarrow{F}(1,n), \dots, \overrightarrow{F}(1,n-j), \dots,, \overrightarrow{F}(1,n-r+2)) = \varepsilon.C(n), \tag{4.9}$$

where $\varepsilon = -1$ or $+1$. It is equivalent to perform a permutation of the vectors column $\overrightarrow{F}(1,n-j)$ in the determinant defining the Casoratian. This permutation is a cycle defined by $\sigma_r = \tau_{1,2} \circ \tau_{2,3} \circ \cdots \circ \tau_{j,j+1} \circ \cdots \circ \tau_{r-1,r}$, where $\tau_{i,j}$ $(i \neq j)$ is the transposition that permutes i and j. A straightforward computation, using $F_n^{(1)} = F_{n-1}^{(r)} = F_{n-1}$, gives

$$det(\widetilde{\mathcal{C}}(n)) = \varepsilon(\sigma_r)C(n), \tag{4.10}$$

where $\varepsilon(\sigma_r) = (-1)^{r-1}$ is the signature of $\sigma_r \in \mathcal{S}_r$ and $\widetilde{\mathcal{C}}(n)$ is the matrix given by

$$\widetilde{\mathcal{C}}(n) = [\overrightarrow{F}(n), \overrightarrow{F}(n-1), \dots, \overrightarrow{F}(n-j), \dots, \overrightarrow{F}(n-r+1)], \tag{4.11}$$

called the *Cassini matrix* and whose entries are given by $\widetilde{\mathcal{C}}_{i,k}^{(n)} = F_{n-k+i}$. Summarizing, the preceding argument allows us to formulate the following general result.

Theorem 4.1. *Let $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$ be the Fibonacci fundamental system and $C(n)$ its related Casoratian. Then, the associated generalized Cassini identity of order $r \geq 3$ is given by*

$$det([\overrightarrow{F}(n), \overrightarrow{F}(n-1), \ldots, \overrightarrow{F}(n-r+1)]) \quad = \quad (-1)^{(n+1)(r-1)} \tag{4.12}$$

for every $n \geq r-2$. Moreover, the generalized Cassini identity of order r can be formulated as follows:

$$\sum_{\sigma\in\mathcal{S}_r} \varepsilon(\sigma) F_{n-\sigma(1)+1} \cdots F_{n-\sigma(r)+r} = \varepsilon(\sigma_r)C(n) = (-1)^{(n+1)(r-1)}, \tag{4.13}$$

where $\mathcal{S}_r$ is the group of permutations of $\{1, 2, \ldots, r\}$, $\varepsilon(\sigma)$ is the signature of $\sigma \in \mathcal{S}_r$, and $\sigma_r = \tau_{1,2} \circ \tau_{2,3} \circ \cdots \circ \tau_{j,j+1} \circ \cdots \circ \tau_{r-1,r}$, with $\tau_{i,j}$ $(i \neq j)$ is the transposition.

Proof. Following the preceding discussions, we have established identity (4.10), namely,

$$det(\widetilde{\mathcal{C}}(n)) = det[\overrightarrow{F}(n), \overrightarrow{F}(n-1), \ldots, \overrightarrow{F}(n-j), \ldots, \overrightarrow{F}(n-r+1)] = \varepsilon(\sigma_r)C(n).$$

On the other hand, for Theorem 2.3, we have

$$C(n) = det[\widetilde{C}(n)] = det[\mathbb{A}^n] = det[\mathbb{A}]^n = (-1)^{n(r-1)} \neq 0.$$

Recall that the determinant of every square matrix $M = (a_{ij})_{1\leq i,j\leq r}$ of order r can be expressed in terms of the permutation group $\mathcal{S}_r$ of $\{1, 2, \ldots, r\}$ as follows:

$$det[M] = \sum_{\sigma\in\mathcal{S}_r} \varepsilon(\sigma) a_{1\sigma(1)} \cdots a_{r\sigma(r)},$$

where $\varepsilon(\sigma)$ is the signature of $\sigma \in \mathcal{S}_r$. For

$$M = \widetilde{C}(n) = [\overrightarrow{F}(n), \overrightarrow{F}(n-1), \ldots, \overrightarrow{F}(n-j), \ldots, \overrightarrow{F}(n-r+1)],$$

we have entries $a_{ij}^{(n)} = F_{n+i-j}$. Hence, for every $\sigma \in \mathcal{S}_r$, $a_{i\sigma(i)}^{(n)} = F_{n+i-\sigma(i)}$. Therefore, we obtain the result

$$det[\widetilde{C}(n)] = \sum_{\sigma\in\mathcal{S}_r} \varepsilon(\sigma) F_{n-\sigma(1)+1} \cdots F_{n-\sigma(r)+r} = \varepsilon(\sigma_r)C(n) = (-1)^{(n+1)(r-1)}.$$

□

Theorem 4.1 shows that the generalized Cassini identity is expressed in terms of the Casoratian of the Fibonacci fundamental system. Moreover, in the determinant form of the generalized Cassini Identity, only the usual generalized Fibonacci numbers $\{F_n\}_{n\geq 0}$ defined by $F_n = F_n^{(r)} = F_{n+1}^{(1)}$ appear.

Generalized Cassini identity for $r = 2$. Let $\mathcal{S}_2$ be the group of permutations of $\{1, 2\}$ with two elements, identity e, and the transposition $\sigma_2 = \tau_{1,2}$. Then, we have $\varepsilon(e) = 1$ and $\varepsilon(\sigma_2) = (-1)^1 = -1$. Application of Theorem 4.1 shows that

$$\begin{aligned} det([\overrightarrow{F}(n), \overrightarrow{F}(n-1)]) &= \sum_{\sigma\in\mathcal{S}_2} \varepsilon(\sigma) F_{n-\sigma(1)+1} F_{n-\sigma(2)+2} \\ &= \varepsilon(e) F_n F_n + \varepsilon(\sigma_2) F_{n-1} F_{n+1} = F_n^2 - F_{n-1}F_{n+1}. \end{aligned} \tag{4.14}$$

On the other hand, we have $det([\overrightarrow{F}(n), \overrightarrow{F}(n-1)) = \varepsilon(\sigma_2) \cdot C(n) = (-1)^{(n+1)}$. Therefore, we obtain

$$det([\overrightarrow{F}(n), \overrightarrow{F}(n-1)]) = F_n^2 - F_{n-1}F_{n+1} = (-1)^{(n+1)}.$$

We see that we can recover expression (4.2), namely, $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$.

Generalized Cassini identity for $r = 3$. Let $\mathcal{S}_3$ be the group of permutations of $\{1, 2, 3\}$ with six elements, identity e, and transpositions $\tau_{1,2}$, $\tau_{1,3}$, $\tau_{2,3}$, $\tau_{1,3} \circ \tau_{2,3}$, $\sigma_3 = \tau_{1,2} \circ \tau_{2,3}$. Then, we have the signatures $\varepsilon(e) = 1$; $\varepsilon(\tau_{1,2}) = \varepsilon(\tau_{1,3}) = \varepsilon(\tau_{2,3}) = (-1)^1 = -1$, and $\varepsilon(\tau_{1,3} \circ \tau_{2,3}) = \varepsilon(\sigma_2) = (-1)^2 = 1$. Application of Theorem 4.1 shows that

$$det([\overrightarrow{F}(n), \overrightarrow{F}(n-1), \overrightarrow{F}(n-2)]) \quad = \quad \varepsilon(\sigma_3).C(n) = (-1)^2(-1)^{n(3-1)} = 1. \qquad (4.15)$$

A long straightforward computation shows that the preceding expression takes the form

$$\begin{aligned} det([\overrightarrow{F}(n), \overrightarrow{F}(n-1), \overrightarrow{F}(n-2)]) &= \sum_{\sigma \in \mathcal{S}_3} \varepsilon(\sigma) F_{n-\sigma(1)+1} F_{n-\sigma(2)+2} F_{n-\sigma(3)+3} \\ &= F_nF_nF_n - F_{n-1}F_{n+1}F_n - F_{n-2}F_nF_{n+2} - F_nF_{n-1}F_{n+1} \\ &\quad + F_{n-2}F_{n+1}F_{n+1} + F_{n-1}F_{n-1}F_{n+2}. \end{aligned}$$

Therefore, expression (4.15) results in the identity

$$F_n(F_n^2 - F_{n+1}F_{n-1}) + F_{n-2}(F_{n+1}^2 - F_{n+2}F_n) - 3F_{n+1}F_nF_{n-1} = 1. \qquad (4.16)$$

Generalized Cassini identity for $r = 4$. Similarly, for $r = 4$ we have the action of the 24 elements of the permutation group $\mathcal{S}_4$. Application of Theorem 4.1 shows that $det([\overrightarrow{F}(n), \overrightarrow{F}(n-1), \overrightarrow{F}(n-2), \overrightarrow{F}(n-3)]) = \varepsilon(\sigma_4) \cdot C(n) = \varepsilon(\sigma_4) \cdot (-1)^{3n}$, where $\varepsilon(\sigma_4) = (-1)^3$ is the signature of $\sigma_4 = \tau_{1,2} \circ \tau_{2,3} \circ \tau_{3,4}$. Thus, this expression takes the form

$$det([\overrightarrow{F}(n), \ldots, \overrightarrow{F}(n-3)]) = \sum_{\sigma \in \mathcal{S}_4} \varepsilon(\sigma) F_{n-\sigma(1)+1} \cdots F_{n-\sigma(4)+4} = (-1)^{n+1}. \qquad (4.17)$$

4.2. **Generalized Cassini Identity and Combinatorics.** The combinatorial properties of the generalized Fibonacci numbers identities (3.6) and (3.7) and those of Cassini's identity can lead to some other identities between the $\rho(n, r)$. More precisely, expression (3.10) (see Proposition 3.3) and Theorem 4.1 permit to us to find new Fibonacci combinatorial identities.

Theorem 4.2. *Let $\mathfrak{F}_r = \{\{F_n^{(s)}\}_{n \geq 0};\ 1 \leq s \leq r\}$ be the Fibonacci fundamental system. Then, the generalized Cassini identity of order r takes the following combinatorial form.*

$$\sum_{\sigma \in \mathcal{S}_r} \varepsilon(\sigma) \left[\prod_{i=1}^{r} \rho(n - \sigma(i) + i + 1, r) \right] = (-1)^{(n+1)(r-1)}, \qquad (4.18)$$

where $\mathcal{S}_r$ is the group of permutations of the set $\{1, 2, \ldots, r\}$ and $\varepsilon(\sigma)$ is the signature of $\sigma \in \mathcal{S}_r$.

Proof. A substitution of expression (3.17) in identity (4.13) of Theorem 4.1 allow us to obtain expression (4.18). □

Specially, for $r = 2$, the usual Cassini identity is given by

$$\rho(n+2,2)\rho(n,r) - \rho(n+1,r)^2 = (-1)^n. \tag{4.19}$$

The generalized combinatorial Cassini identity of order $r = 3$ is given by

$$\begin{aligned} &\rho(n+1,3)\left(\rho(n+1,3)^2 - \rho(n+2,3)\rho(n,3)\right) + \rho(n+2,3)^2\rho(n-1,3) \\ &-\rho(n+3,3)\rho(n+1,3)\rho(n-1,3) - 3\rho(n+2,3)\rho(n+1,3)\rho(n,3) = 1. \end{aligned} \tag{4.20}$$

To the best of our knowledge, Theorems 4.1 and 4.2 are not in the current literature, regarding the combinatorial aspect related to generalized Fibonacci numbers.

5. Concluding Remarks and Perspective

The results obtained in this study show the importance of the Casoratian matrix and Casoratian method in connection with the companion matrix. For expression (1.1) considered as the difference equation, the two basic solutions play an important role, for the fundamental Fibonacci system and for the generalization of Cassini's identity.

Our method can also give similar results for other generalizations of the Fibonacci numbers or Pell numbers. Some preliminary results have been obtained in this direction.

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