ON ORESME NUMBERS AND THEIR CONNECTION WITH FIBONACCI AND PELL NUMBERS

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Abstract. In this paper, we consider determinants for some families of Toeplitz-Hessenberg matrices whose entries are Oresme numbers. These determinant formulas may also be rewritten as identities involving sums of products of Oresme numbers and multinomial coefficients. In particular, we establish a connection between the Oresme and the Fibonacci and Pell sequences via Toeplitz-Hessenberg determinants.

1. Introduction and Preliminaries

In [3], Horadam presented a history of the number sequence

$$\{O_n\}_{n\geq 1} = \left\{\frac{n}{2^n}\right\}_{n\geq 1}, \tag{1.1}$$

attributed to the famous French naturalist and philosopher Nicole Oresme (1323–1382).

The Oresme numbers O_n also can be defined by the recurrence

$$O_n = O_{n-1} - \frac{1}{4}O_{n-2}, \quad n \geq 2, \tag{1.2}$$

with initial values $O_0 = 0$, $O_1 = \frac{1}{2}$.

Many properties of the Oresme numbers are given in [2, 3].

We study some families of Toeplitz-Hessenberg determinants whose entries are Oresme numbers. Recall that a *Toeplitz-Hessenberg matrix* is an $n \times n$ matrix of the form

$$M_n(a_0; a_1, a_2, \ldots, a_n) = \begin{bmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{bmatrix}, \tag{1.3}$$

where $a_0 \neq 0$ and $a_k \neq 0$ for at least one $k > 0$.

Expanding the determinant $\det(M_n)$ according to the last row repeatedly, we obtain the recurrence

$$\det(M_n) = \sum_{k=1}^{n}(-a_0)^{k-1}a_k \det(M_{n-k}), \tag{1.4}$$

where, by definition, $\det(M_0) = 1$.

The following result is known as *Trudi's formula* [4]. It gives the multinomial extension of $\det(M_n)$ as follows:

$$\det(M_n) = \sum_{\substack{s_1,\ldots,s_n\geq 0 \\ s_1+2s_2+\cdots+ns_n=n}} (-a_0)^{n-(s_1+\cdots+s_n)} \frac{(s_1+\cdots+s_n)!}{s_1!\cdots s_n!} a_1^{s_1}a_2^{s_2}\cdots a_n^{s_n}. \tag{1.5}$$

We investigate particular cases of the Toeplitz-Hessenberg matrix (1.3) with $a_0 = \pm 1$. For brevity, we write $D_{\pm}(a_1, a_2, \ldots, a_n)$ in place of $\det\big(M_n(\pm 1; a_1, a_2, \ldots, a_n)\big)$.

2. Connection Formulas Between the Oresme Numbers with Fibonacci and Pell Numbers

In the next two sections, we consider determinants of some Toeplitz-Hessenberg matrices with Oresme number entries. Our first result provides a relation between the Oresme numbers and the Fibonacci and Pell sequences.

Recall the *Fibonacci sequence* $\{F_n\}_{n\geq 0}$ is defined by the initial values $F_0 = 0$, $F_1 = 1$, and the recurrence $F_n = F_{n-1} + F_{n-2}$, where $n \geq 2$. The *Pell sequence* $\{P_n\}_{n\geq 0}$ is defined by the recurrence $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$ with $P_0 = 0$ and $P_1 = 1$ (see, for example, [1]).

Theorem 2.1. *For $n \geq 1$,*

$$D_-(O_1, O_2, \ldots, O_n) = \frac{F_{2n}}{2^n}, \tag{2.1}$$

$$D_-(O_1, O_3, \ldots, O_{2n-1}) = \frac{F_{3n-1}}{2^{2n-1}},$$

$$D_-(O_0, O_1, \ldots, O_{n-1}) = \frac{P_{n-1}}{2^{n-1}}.$$

Proof. We will prove formula (2.1) using induction on n. The other proofs follow similarly, so we omit them. Let

$$D_n = D_-(O_1, O_2, \ldots, O_n).$$

Using the Oresme numbers in (1.3) shows that (2.1) holds for $n = 1$ and $n = 2$. Suppose it is true for all positive integers $k \leq n - 1$, where $n \geq 2$.

Using recurrences (1.4) and (1.2), we have

$$\begin{aligned} D_n &= \sum_{i=1}^{n} O_i D_{n-i} \\ &= O_1 D_{n-1} + \sum_{i=2}^{n} \left(O_{i-1} - \frac{1}{4} O_{i-2} \right) D_{n-i} \\ &= \frac{1}{2} D_{n-2} + \sum_{i=1}^{n-1} O_i D_{n-i-1} - \frac{1}{4} \sum_{i=0}^{n-2} O_i D_{n-i-2} \\ &= \frac{1}{2} D_{n-1} + D_{n-1} - \frac{1}{4} D_{n-2} \\ &= \frac{3}{2} D_{n-1} - \frac{1}{4} D_{n-2}. \end{aligned}$$

Using the induction hypothesis, we obtain

$$\begin{aligned} D_n &= \frac{3}{2} \cdot \frac{F_{2n-2}}{2^{n-1}} - \frac{1}{4} \cdot \frac{F_{2n-4}}{2^{n-2}} \\ &= \frac{1}{2^n} \left(3F_{2n-2} - F_{2n-4} \right) = \frac{F_{2n}}{2^n}. \end{aligned}$$

Consequently, formula (2.1) is true for n. Therefore, by induction, the formula holds for all positive integers n. ☐

3. Some Toeplitz-Hessenberg Determinants with Oresme Number Entries

In this section, we evaluate some Toeplitz-Hessenberg determinants with special Oresme number entries.

Theorem 3.1. *For $n \geq 1$, the following identities hold.*

$$D_+(O_0, O_1, \dots, O_{n-1}) = \frac{(-1)^{\lfloor n+\frac{1}{2}\rfloor}\sqrt{2}i}{4}\left(\left(\frac{1}{2}+\frac{\sqrt{2}}{2}i\right)^{n-1}-\left(\frac{1}{2}-\frac{\sqrt{2}}{2}i\right)^{n-1}\right),$$

$$D_+(O_0, O_2, \dots, O_{2n-2}) = \frac{\sqrt{2}i}{4}\left(\left(-\frac{1}{4}+\frac{\sqrt{2}}{2}i\right)^{n-1}-\left(-\frac{1}{4}-\frac{\sqrt{2}}{2}i\right)^{n-1}\right),$$

$$D_-(O_0, O_2, \dots, O_{2n-2}) = \frac{\sqrt{2}}{4}\left(\left(\frac{1}{4}+\frac{\sqrt{2}}{2}\right)^{n-1}-\left(\frac{1}{4}-\frac{\sqrt{2}}{2}\right)^{n-1}\right),$$

$$D_+(O_1, O_2, \dots, O_n) = \frac{n-3\lfloor\frac{n+1}{3}\rfloor}{2^n},$$

$$D_+(O_1, O_3, \dots, O_{2n-1}) = \frac{(-1)^{\lfloor\frac{n}{2}\rfloor}3^{\lfloor\frac{n-1}{2}\rfloor}}{2^{2n-1}}, \tag{3.1}$$

$$D_+(O_2, O_3, \dots, O_{n+1}) = (-1)^{\lfloor\frac{5n}{4}\rfloor}\frac{\lfloor\frac{n+1}{4}\rfloor-\lceil\frac{n+1}{4}\rceil}{2^{\lfloor 3n/2\rfloor}},$$

$$D_+(O_2, O_4, \dots, O_{2n}) = (-1)^{\lfloor n/2\rfloor}\frac{1-(-1)^n}{4^n},$$

$$D_-(O_2, O_4, \dots, O_{2n}) = \frac{\sqrt{3}}{3}\left(\left(\frac{1}{2}+\frac{\sqrt{3}}{4}\right)^{n}-\left(\frac{1}{2}-\frac{\sqrt{3}}{4}\right)^{n}\right), \tag{3.2}$$

$$D_+(O_3, O_4, \dots, O_{n+2}) = \frac{2\sqrt{7}i}{7}\left(\left(\frac{1}{4}+\frac{\sqrt{7}}{4}i\right)^{3n+1}-\left(\frac{1}{4}-\frac{\sqrt{7}}{4}i\right)^{3n+1}\right),$$

$$D_+(O_3, O_5, \dots, O_{2n+1}) = \frac{512\sqrt{7}i}{7}\left(\left(-\frac{1}{16}-\frac{\sqrt{7}}{16}i\right)^{n+3}-\left(-\frac{1}{16}+\frac{\sqrt{7}}{16}i\right)^{n+3}\right),$$

$$D_+(O_4, O_6, \dots, O_{2n+2}) = (-1)^{\lfloor 5n/4\rfloor}\frac{\lfloor\frac{n+1}{4}\rfloor-\lceil\frac{n+1}{4}\rceil}{2^{\lfloor 5n/2\rfloor}},$$

where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ are the floor function and the ceiling function, respectively; $i=\sqrt{-1}$.

Proof. We will prove only (3.1) and (3.2), the other ones can be proved in the same way.

Proof of (3.1). Let $D_n = D_+(O_1, O_3, \dots, O_{2n-1})$. Using the Oresme numbers in (1.3), it follows that

$$D_n = \begin{cases} \dfrac{(-1)^{\frac{n}{2}}3^{\frac{n}{2}-1}}{2^{2n-1}}, & \text{if } n \text{ is even;} \\ \dfrac{(-3)^{\frac{n-1}{2}}}{2^{2n-1}}, & \text{otherwise,} \end{cases} \tag{3.3}$$

which can be established using induction.

When $n = 1$ and $n = 2$, the formula holds. Assuming (3.3) holds for $n-1$, we now prove it for $n \geq 2$. Let n be even. The proof for the case when n is odd is similar. Using (1.1), (1.4), and (3.3), we have

$$\begin{aligned}
D_n &= \sum_{i=1}^{n} (-1)^{i-1} O_{2i-1} D_{n-i} \\
&= \sum_{\substack{i=2 \\ (i \text{ is even})}}^{n-2} (-1)^{i-1} O_{2i-1} D_{n-i} + \sum_{\substack{i=1 \\ (i \text{ is odd})}}^{n-1} (-1)^{i-1} O_{2i-1} D_{n-i} + (-1)^{n-1} O_{2n-1} D_0 \\
&= \sum_{k=1}^{\frac{n}{2}-1} (-1)^{2k-1} \frac{4k-1}{2^{4k-1}} D_{n-2k} + \sum_{k=1}^{\frac{n}{2}} (-1)^{2k-2} \frac{4k-3}{2^{4k-3}} D_{n-2k+1} - \frac{2n-1}{2^{2n-1}} \\
&= \sum_{k=1}^{\frac{n}{2}-1} \frac{4k-1}{2^{4k-1}} \frac{(-3)^{\frac{n-2k}{2}-1}}{2^{2(n-2k)-1}} + \sum_{k=1}^{\frac{n}{2}} \frac{4k-3}{2^{4k-3}} \frac{(-3)^{\frac{n-2k}{2}}}{2^{2(n-2k+1)-1}} - \frac{2n-1}{2^{2n-1}} \\
&= \frac{(-3)^{\frac{n}{2}}}{2^{2n-2}} \left(\sum_{k=1}^{\frac{n}{2}-1} \frac{4k-1}{(-3)^{k+1}} + \sum_{k=1}^{\frac{n}{2}} \frac{4k-3}{(-3)^k} \right) - \frac{2n-1}{2^{2n-1}} \\
&= \frac{(-3)^{\frac{n}{2}}}{3 \cdot 2^{2n-5}} \left(\sum_{k=1}^{\frac{n}{2}-1} \frac{k}{(-3)^k} - \sum_{k=1}^{\frac{n}{2}-1} \frac{1}{(-3)^k} \right) + \frac{2n-5}{2^{2n-1}}. \qquad (3.4)
\end{aligned}$$

Using the geometric series, it can be seen that the following two sums hold.

$$\sum_{j=1}^{p} \frac{(-1)^j}{3^j} = \frac{1}{4(-3)^p} - \frac{1}{4},$$

$$\sum_{j=1}^{p} \frac{(-1)^j j}{3^j} = \frac{p}{4(-3)^p} + \frac{3}{16(-3)^p} - \frac{3}{16}.$$

From (3.4), using the formulas above, we obtain

$$\begin{aligned}
D_n &= \frac{(-3)^{\frac{n}{2}}}{3 \cdot 2^{2n-5}} \left(\frac{\frac{n}{2}-1}{4 \cdot (-3)^{\frac{n}{2}-1}} + \frac{3}{16 \cdot (-3)^{\frac{n}{2}-1}} - \frac{3}{16} - \frac{1}{4 \cdot (-3)^{\frac{n}{2}-1}} + \frac{1}{4} \right) + \frac{2n-5}{2^{2n-1}} \\
&= -\frac{n}{2^{2n-2}} + \frac{5}{2^{2n-1}} + \frac{(-3)^{\frac{n}{2}}}{3 \cdot 2^{2n-1}} + \frac{2n}{2^{2n-1}} - \frac{5}{2^{2n-1}} \\
&= \frac{(-3)^{\frac{n}{2}}}{3 \cdot 2^{2n-1}}.
\end{aligned}$$

Since formula (3.1) holds for n, it follows by induction that it is true for all positive integers n.

Proof of (3.2). Let $D_n = D_-(O_2, O_4, \ldots, O_{2n})$. Formula (3.2) clearly holds when $n = 1$ and $n = 2$. Suppose it is true for all $k \leq n-1$, where $n \geq 3$. Using (1.4) and recurrence

$$O_k = \frac{1}{2} O_{k-2} - \frac{1}{16} O_{k-4}, \quad k \geq 4,$$

we have

$$\begin{aligned}
D_n &= \sum_{i=1}^{n} O_{2i} D_{n-i} \\
&= O_2 D_{n-1} + \sum_{i=2}^{n-1} \left(\frac{1}{2} O_{2i-2} - \frac{1}{16} O_{2i-4} \right) D_{n-i} + O_{2n} D_0 \\
&= \frac{1}{2} D_{n-1} + \frac{1}{2} \sum_{i=1}^{n-2} O_{2i} D_{n-i-1} - \frac{1}{16} \sum_{i=1}^{n-3} O_{2i} D_{n-i-2} + \frac{2n}{2^{2n}} \\
&= \frac{1}{2} D_{n-1} + \frac{1}{2} \left(\sum_{i=1}^{n-1} O_{2i} D_{n-i-1} - O_{2(n-1)} D_0 \right) \\
&\quad - \frac{1}{16} \left(\sum_{i=1}^{n-2} O_{2i} D_{n-i-2} - O_{2(n-2)} D_0 \right) + \frac{2n}{2^{2n}} \\
&= \frac{1}{2} D_{n-1} + \frac{1}{2} \left(D_{n-1} - O_{2(n-1)} \right) - \frac{1}{16} \left(D_{n-2} - O_{2(n-2)} \right) + \frac{2n}{2^{2n}} \\
&= D_{n-1} - \frac{1}{16} D_{n-2}.
\end{aligned}$$

Using the induction hypothesis, we obtain

$$\begin{aligned}
D_n &= \frac{\sqrt{3}}{3} \left(\left(\frac{1}{2} + \frac{\sqrt{3}}{4} \right)^{n-1} - \left(\frac{1}{2} - \frac{\sqrt{3}}{4} \right)^{n-1} \right) - \frac{\sqrt{3}}{48} \left(\left(\frac{1}{2} + \frac{\sqrt{3}}{4} \right)^{n-2} - \left(\frac{1}{2} - \frac{\sqrt{3}}{4} \right)^{n-2} \right) \\
&= \frac{\sqrt{3}}{3} \left(\frac{1}{2} + \frac{\sqrt{3}}{4} \right)^{n-2} \left(\frac{7}{16} + \frac{\sqrt{3}}{4} \right) - \frac{\sqrt{3}}{3} \left(\frac{1}{2} - \frac{\sqrt{3}}{4} \right)^{n-2} \left(\frac{7}{16} - \frac{\sqrt{3}}{4} \right) \\
&= \frac{\sqrt{3}}{3} \left(\left(\frac{1}{2} + \frac{\sqrt{3}}{4} \right)^{n} - \left(\frac{1}{2} - \frac{\sqrt{3}}{4} \right)^{n} \right).
\end{aligned}$$

Consequently, formula (3.2) is true in the n case and thus, by induction, it holds for all positive integers. □

4. Multinomial Extension of Toeplitz-Hessenberg Determinants

In this section, we focus on the multinomial extension of Theorems 2.1 and 3.1, using Trudi's formula (1.5).

Corollary 4.1. *For $n \geq 1$, the following formulas hold.*

$$F_{2n} = 2^n \cdot \sum_{\substack{s_1,\ldots,s_n \geq 0 \\ s_1+2s_2+\cdots+ns_n=n}} \frac{(s_1+\cdots+s_n)!}{s_1! \cdots s_n!} O_1^{s_1} O_2^{s_2} \cdots O_n^{s_n}, \tag{4.1}$$

$$F_{3n-1} = 2^{2n-1} \cdot \sum_{\substack{s_1,\ldots,s_n \geq 0 \\ s_1+2s_2+\cdots+ns_n=n}} \frac{(s_1+\cdots+s_n)!}{s_1! \cdots s_n!} O_1^{s_1} O_3^{s_2} \cdots O_{2n-1}^{s_n}, \tag{4.2}$$

$$P_{n-1} = 2^{n-1} \cdot \sum_{\substack{s_1,\ldots,s_{n-1}\geq 0\\ 2s_1+3s_2+\cdots+ns_{n-1}=n}} \frac{(s_1+\cdots+s_{n-1})!}{s_1!\cdots s_{n-1}!} O_1^{s_1} O_2^{s_2}\cdots O_{n-1}^{s_{n-1}}. \tag{4.3}$$

Corollary 4.2. *Let $n \geq 1$, $\sigma_n = s_1 + \cdots + s_n$, and let $m_n(s) = \frac{(s_1+\cdots+s_n)!}{s_1!\cdots s_n!}$ denote the multinomial coefficient. Then,*

$$\sum_{\substack{s_1,\ldots,s_{n-1}\geq 0\\ 2s_1+\cdots+ns_{n-1}=n}} (-1)^{\sigma_{n-1}} m_{n-1}(s) O_1^{s_1} O_2^{s_2}\cdots O_{n-1}^{s_{n-1}}$$
$$= \frac{(-1)^{\lfloor n+\frac{1}{2}\rfloor} i}{2\sqrt{2}} \left(\left(-\frac{1}{2}-\frac{\sqrt{2}}{2}i\right)^{n-1} - \left(-\frac{1}{2}+\frac{\sqrt{2}}{2}i\right)^{n-1}\right),$$

$$\sum_{\substack{s_1,\ldots,s_n\geq 0\\ s_1+2s_2+\cdots+ns_n=n}} m_n(s) O_1^{s_1} O_2^{s_2}\cdots O_n^{s_n} = \frac{n-3\left\lfloor\frac{n+1}{3}\right\rfloor}{(-2)^n},$$

$$\sum_{\substack{s_1,\ldots,s_n\geq 0\\ s_1+2s_2+\cdots+ns_n=n}} (-1)^{\sigma_n} m_n(s) O_2^{s_1} O_3^{s_2}\cdots O_{n+1}^{s_n} = (-1)^{\lfloor n/4\rfloor} \frac{\left\lfloor\frac{n+1}{4}\right\rfloor - \left\lceil\frac{n+1}{4}\right\rceil}{2^{\lfloor 3n/2\rfloor}},$$

$$\sum_{\substack{s_1,\ldots,s_n\geq 0\\ s_1+2s_2+\cdots+ns_n=n}} (-1)^{\sigma_n} m_n(s) O_3^{s_1} O_4^{s_2}\cdots O_{n+2}^{s_n}$$
$$= \frac{2\sqrt{7}i}{7}\left(\left(-\frac{1}{4}+\frac{\sqrt{7}}{4}i\right)^{3n+1} - \left(-\frac{1}{4}-\frac{\sqrt{7}}{4}i\right)^{3n+1}\right),$$

$$\sum_{\substack{s_1,\ldots,s_n\geq 0\\ s_1+2s_2+\cdots+ns_n=n}} m_n(s) O_1^{s_1} O_3^{s_2}\cdots O_{2n-1}^{s_n} = \frac{(-1)^{\left\lfloor\frac{3n}{2}\right\rfloor} 3^{\left\lfloor\frac{n-1}{2}\right\rfloor}}{2^{2n-1}},$$

$$\sum_{\substack{s_1,\ldots,s_n\geq 0\\ s_1+2s_2+\cdots+ns_n=n}} (-1)^{\sigma_n} m_n(s) O_3^{s_1} O_5^{s_2}\cdots O_{2n+1}^{s_n}$$
$$= \frac{512\sqrt{7}i}{7}\left(\left(\frac{1}{16}-\frac{\sqrt{7}}{16}i\right)^{n+3} - \left(\frac{1}{16}+\frac{\sqrt{7}}{16}i\right)^{n+3}\right),$$

$$\sum_{\substack{s_1,\ldots,s_{n-1}\geq 0\\ 2s_1+\cdots+ns_{n-1}=n}} (-1)^{\sigma_{n-1}} m_{n-1}(s) O_2^{s_1} O_4^{s_2}\cdots O_{2n-2}^{s_{n-1}}$$
$$= \frac{\sqrt{2}i}{4}\left(\left(\frac{1}{4}+\frac{\sqrt{2}}{2}i\right)^{n-1} - \left(\frac{1}{4}-\frac{\sqrt{2}}{2}i\right)^{n-1}\right),$$

$$\sum_{\substack{s_1,\ldots,s_{n-1}\geq 0\\ 2s_1+\cdots+ns_{n-1}=n}} m_{n-1}(s) O_2^{s_1} O_4^{s_2}\cdots O_{2n-2}^{s_{n-1}} = \frac{\sqrt{2}}{4}\left(\left(\frac{1}{4}+\frac{\sqrt{2}}{2}\right)^{n-1} - \left(\frac{1}{4}-\frac{\sqrt{2}}{2}\right)^{n-1}\right),$$

$$\sum_{\substack{s_1,\ldots,s_n\geq 0\\ s_1+2s_2+\cdots+ns_n=n}} (-1)^{\sigma_n} m_n(s) O_2^{s_1} O_4^{s_2}\cdots O_{2n}^{s_n} = (-1)^{\lfloor n/2\rfloor} \frac{(-1)^n - 1}{4^n}, \tag{4.4}$$

$$\sum_{\substack{s_1,\ldots,s_n\geq 0\\ s_1+2s_2+\cdots+ns_n=n}} m_n(s)O_2^{s_1}O_4^{s_2}\cdots O_{2n}^{s_n} = \frac{\sqrt{3}}{3}\left(\left(\frac{1}{2}+\frac{\sqrt{3}}{4}\right)^n - \left(\frac{1}{2}-\frac{\sqrt{3}}{4}\right)^n\right),$$

$$\sum_{\substack{s_1,\ldots,s_n\geq 0\\ s_1+2s_2+\cdots+ns_n=n}} (-1)^{\sigma_n} m_n(s)O_4^{s_1}O_6^{s_2}\cdots O_{2n+2}^{s_n} = (-1)^{\lfloor n/4\rfloor}\frac{\lfloor\frac{n+1}{4}\rfloor - \lceil\frac{n+1}{4}\rceil}{2^{\lfloor 5n/2\rfloor}},$$

where F_n and P_n are the nth Fibonacci and Pell numbers, respectively.

Example 4.3. *Formula* (4.1) *yields*

$$\begin{aligned} F_6 &= 2^3\cdot \sum_{\substack{s_1,s_2,s_3\geq 0\\ s_1+2s_2+3s_3=3}} \frac{(s_1+s_2+s_3)!}{s_1!s_2!s_3!}O_1^{s_1}O_2^{s_2}O_3^{s_3}\\ &= 8(O_1^3+2O_1O_2+O_3). \end{aligned}$$

Similarly, it follows from (4.3) *and* (4.4)*, respectively, that*

$$32(O_1^3+2O_1O_3+O_2^2+O_5) = P_5,$$
$$O_2^4 - 3O_2^3O_4 + 2O_2O_6 + O_4^2 - O_8 = 0.$$

After simple manipulations using (4.1), (4.2), and (4.3), respectively, we obtain the following.

Corollary 4.4. *For $n \geq 1$, the following formulas hold.*

$$F_{2n} = \sum_{\substack{s_1,\ldots,s_n\geq 0\\ s_1+2s_2+\cdots+ns_n=n}} m_n(s)\,1^{s_1}2^{s_2}\cdots n^{s_n},$$

$$F_{3n-1} = 2^{n-1}\cdot \sum_{\substack{s_1,\ldots,s_n\geq 0\\ s_1+2s_2+\cdots+ns_n=n}} m_n(s)\left(\frac{1}{1}\right)^{s_1}\left(\frac{3}{2}\right)^{s_2}\cdots\left(\frac{2n-1}{2^{n-1}}\right)^{s_n},$$

$$P_{n-1} = 2^{n-1}\cdot \sum_{\substack{s_1,\ldots,s_{n-1}\geq 0\\ 2s_1+3s_2+\cdots+ns_{n-1}=n}} m_{n-1}(s)\left(\frac{1}{2}\right)^{s_1}\left(\frac{2}{4}\right)^{s_2}\cdots\left(\frac{n-1}{2^{n-1}}\right)^{s_{n-1}}.$$

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