CENTRAL LIMIT THEOREMS FOR GAPS OF GENERALIZED ZECKENDORF DECOMPOSITIONS

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Abstract. Zeckendorf proved that every integer can be written uniquely as a sum of nonadjacent Fibonacci numbers $\{1, 2, 3, 5, \ldots\}$. This has been extended to many other recurrence relations $\{G_n\}$ (with their own notion of a legal decomposition). It has also been proved that the distribution of the number of summands of an $M \in [G_n, G_{n+1})$ converges to a Gaussian as $n \to \infty$. We prove that for any nonnegative integer g, the average number of gaps of size g in many generalized Zeckendorf decompositions is $C_\mu n + d_\mu + o(1)$ for constants $C_\mu > 0$ and d_μ depending on g and the recurrence, the variance of the number of gaps of size g is similarly $C_\sigma n + d_\sigma + o(1)$ for constants $C_\sigma > 0$ and d_σ, and the number of gaps of size g of an $M \in [G_n, G_{n+1})$ converges to a Gaussian as $n \to \infty$. We show this by proving a general result on when an associated two-dimensional recurrence converges to a Gaussian, and additionally re-derive other results in the literature.

1. Introduction

1.1. **Previous Results.** Zeckendorf [27] proved that if the Fibonacci numbers are defined by $F_1 = 1, F_2 = 2$, and $F_{n+1} = F_n + F_{n-1}$, then every integer can be written as a sum of nonadjacent terms. The standard proof is by the greedy algorithm, although combinatorial approaches exist (see [16]). More generally, one can consider other sequences of numbers and rules for a legal decomposition, and ask when a unique decomposition exists, and if it does how the summands are distributed.

There has been much work on these decomposition problems. In this paper, we concentrate on decompositions of *positive linear recurrences*, defined below (see [1, 10] for signed decompositions, [9] for f-decompositions, [5, 6, 7] for decompositions of recurrences where the leading term vanishes, and [8] for a lattice based example).

Definition 1.1. *A positive linear recurrence sequence (* PLRS*) is a nonconstant sequence $\{G_n\}$ satisfying*

$$G_n \;=\; c_1 G_{n-1} + \cdots + c_L G_{n-L} \tag{1.1}$$

with nonnegative integer coefficients c_i, with $c_1, c_L, L \geq 1$ and initial conditions $G_1 = 1$ and $G_n \;=\; c_1 G_{n-1} + c_2 G_{n-2} + \cdots + c_{n-1} G_1 + 1$ for $1 < n \leq L$.

Definition 1.2 (Legal decomposition)**.** *For a positive linear recurrence sequence $\{G_n\}$, a* legal *decomposition of an integer $M > 0$ is a decomposition*

$$M \;=\; \sum_{i=1}^{N} a_i G_{N+1-i} \tag{1.2}$$

with $a_1 > 0$ and the other $a_i \geq 0$, and one of the following two conditions holds.

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(1) *$N < L$ and $a_i = c_i$ for $1 \le i \le N$.*
(2) *There exists an $s \in \{1, \ldots, L\}$ such that*
  (i) $a_1 = c_1$, $a_2 = c_2$, ..., $a_{s-1} = c_{s-1}$, and $a_s < c_s$,
  (ii) $a_{s+1}, \ldots, a_{s+\ell} = 0$ for some $\ell \ge 0$, and
  (iii) $\{b_i\}_{i=1}^{N-s-\ell}$ (with $b_i = a_{s+\ell+i}$) is either legal or empty.

Informally, a legal decomposition is one where we cannot use the recurrence relation to replace a linear combination of summands with another summand, and the coefficient of each summand is appropriately bounded. We conclude this section by describing previous results on these sequences (see [12, 13, 14, 15, 19, 17, 22, 23, 24, 25, 26], especially [22] for proofs), and then state our new theorems in Section 1.2.

Theorem 1.3 (Generalized Zeckendorf Theorem, [22])**.** *Let $\{G_n\}$ be a positive linear recurrence sequence. Each integer $M > 0$ admits a unique legal decomposition.*

Given a legal decomposition, we have the following definitions.

Definition 1.4 (Gaps of a decomposition)**.** *Suppose we are given a PLRS $\{G_n\}$ and a legal decomposition*

$$M \;=\; \sum_{i=1}^{N} a_i G_{N+1-i} \;=\; G_{i_1} + G_{i_2} + \cdots + G_{i_k}, \tag{1.3}$$

for some positive integer $k = a_1 + a_2 + \cdots + a_N$ and $i_1 \ge i_2 \ge \cdots \ge i_k$. The gaps *in the decomposition of M are the numbers $i_1 - i_2$, $i_2 - i_3$, ..., $i_{k-1} - i_k$ (for example, $101 = F_{10} + F_5 + F_3 + F_1$, and thus, has gaps 5, 2, and 2).*

Definition 1.5 (Gap random variables)**.** *Given a PLRS $\{G_n\}$ and a positive integer M, we let $k_\Sigma(M)$ denote the number of summands in the decomposition of M and $k_g(M)$ the number of gaps of size g in M's decomposition. Let $K_{\Sigma,n}$ be the random variable equal to $k_\Sigma(M)$ for an M chosen uniformly from $[G_n, G_{n+1})$, and let $K_{g,n}$ be a random variable equal to $k_g(M)$ for an M chosen uniformly from $[G_n, G_{n+1})$. Thus, $k_g(M)$ is a decomposition of $k_\Sigma(M)$, as*

$$k_\Sigma(M) \;=\; 1 + \sum_{g=0}^{\infty} k_g(M). \tag{1.4}$$

The next result concerns the average number of summands in decompositions, generalizing Lekkerkerker's [18] work on this problem for the Fibonacci numbers.

Theorem 1.6 (Generalized Lekkerkerker's Theorem for PLRS, [22])**.** *Let $\{G_n\}$ be a PLRS, let $K_{\Sigma,n}$ be the random variable defined above, and let $\mu_n = \mathbf{E}[K_{\Sigma,n}]$. Then, there exist constants $C_\mu > 0$, d_μ, and $\gamma_\mu \in (0,1)$, depending only on L and the c_i's of the recurrence relation such that*

$$\mu_n \;=\; C_\mu n + d_\mu + O(\gamma_\mu^n). \tag{1.5}$$

Theorem 1.7 (Variance Is Linear for PLRS, [22])**.** *Let $\{G_n\}$ be a PLRS, let $K_{\Sigma,n}$ be the random variable defined above, and let $\sigma_n^2 = \mathbf{Var}[K_{\Sigma,n}]$. Then, there exist constants $C_\sigma > 0$, d_σ, and $\gamma_\sigma \in (0,1)$ depending only on L and the c_i's of the recurrence relation such that*

$$\sigma_n^2 \;=\; C_\sigma n + d_\sigma + O(\gamma_\sigma^n). \tag{1.6}$$

Theorem 1.8 (Gaussian Behavior for Number of Summands in PLRS, [22])**.** *Let $\{G_n\}$ be a PLRS and let $K_{\Sigma,n}$ be the random variable defined above. The mean μ_n and variance σ_n^2 of $K_{\Sigma,n}$ grow linearly in n, and $(K_{\Sigma,n} - \mu_n)/\sigma_n$ converges weakly to the standard normal $N(0,1)$ as $n \to \infty$.*

Much less has been written on $k_g(M)$ and $K_{g,n}$. We show that similar Central Limit results hold for gaps. The techniques we introduce to prove these results allow us to easily prove some results already in the literature such as the three previous theorems.

Beckwith, et al. [2], Bower, et al. [4], and Dorward, et al. [11] explored the distribution of gaps in Generalized Zeckendorf Decompositions arising from PLRS, proving (in the limit as $n \to \infty$) exponential decay in the probability that a gap in the decomposition of $M \in [G_n, G_{n+1})$ has length g as g grows and determining that the distribution of the longest gap between summands behaves similarly to what is seen in the distribution of the longest run of heads in tossing a biased coin. We improve on the first result and establish lower order terms (the previous work had $O(1)$ instead of $d + o(1)$ below), then prove the variance has a similar linear behavior, and finally show Gaussian behavior for fixed g. See [20] for a similar analysis concentrating on the Fibonacci case, where the simplicity of the defining recurrence allows simplifications in the analysis.

1.2. New Results.

We now state our new results.

Theorem 1.9 (Generalized Lekkerkerker's Theorem for Gaps of Decompositions)**.** *Let $g \geq 0$ be a fixed positive integer. Let $\{G_n\}$ be a PLRS with the additional constraint that all c_i's are positive. Suppose there exists $n_0 \in \mathbb{N}$ such that $K_{g,n}$, the random variable defined above, is nontrivial (i.e., is not the constant 0) for $n \geq n_0$. Let $\mu_{g,n} = \mathbf{E}[K_{g,n}]$. Then, there exist constants $C_{\mu,g} > 0$, $d_{\mu,g}$, and $\gamma_{\mu,g} \in (0,1)$ depending only on g, L, and the c_i's of the recurrence relation such that*

$$\mu_{g,n} \ = \ C_{\mu,g} n + d_{\mu,g} + O(\gamma_{\mu,g}^n). \tag{1.7}$$

Theorem 1.10 (Variance Is Linear for Gaps of Decompositions)**.** *Let $g \geq 0$ be a fixed positive integer. Let $\{G_n\}$ be a PLRS with the additional constraint that all c_i's are positive. Suppose there exists $n_0 \in \mathbb{N}$ such that $K_{g,n}$, the random variable defined above, is nontrivial for $n \geq n_0$. Let $\sigma_{g,n}^2 = \mathbf{Var}[K_{g,n}]$. Then, there exists constants $C_{\sigma,g} > 0$, $d_{\sigma,g}$, and $\gamma_{\sigma,g} \in (0,1)$ depending only on g, L, and the c_i's of the recurrence relation such that*

$$\sigma_{g,n}^2 \ = \ C_{\sigma,g} n + d_{\sigma,g} + O(\gamma_{\sigma,g}^n). \tag{1.8}$$

These two theorems are intermediate results in the proof of the next theorem, which is the main result of this paper. The next theorem proves we also obtain Gaussian behavior if we fix the gap size and if that gap size occurs. Note there are never gaps of length 1 between summands in Zeckendorf decompositions arising from Fibonacci numbers, and we must exclude such cases.

Theorem 1.11 (Gaussian Behavior for Gaps of Decompositions)**.** *Let $g \geq 0$ be a fixed positive integer. Let $\{G_n\}$ be a PLRS with the additional constraint that all c_i's are positive. Suppose there exists $n_0 \in \mathbb{N}$ such that $K_{g,n}$, the random variable defined above, is nontrivial for $n \geq n_0$. Then, the mean $\mu_{g,n}$ and variance $\sigma_{g,n}^2$ of $K_{g,n}$ grow linearly in n, and $(K_{g,n} - \mu_{g,n})/\sigma_{g,n}$ converges weakly to the standard normal $N(0,1)$ as $n \to \infty$.*

The key to the proof is Theorem 3.3, which states that certain two-dimensional recursions exhibit Gaussian behavior. Theorem 3.3 generalizes the well-known result that the recurrence $a_{n,k} = a_{n-1,k} + a_{n-1,k-1}$, which produces the binomials $\binom{n}{k}$, produces probability distribution given by $a_{n,0}, \dots, a_{n,n}$ that, when normalized, converge to a normal distribution. Our proof shows that $p_{g,n,k}$, the number of $M \in [G_n, G_{n+1})$ with exactly k gaps of size g, satisfies a homogenous two-dimensional recursion that fits the framework of Theorem 3.3 (see Section 4).

Similar to the work of Miller and Wang [22, 23], we use the method of moments to prove that our random variables converge to Gaussians. More precisely, we prove that the moments of the nth random variable $K_{g,n}$ (or $K_{\Sigma,n}$), $\tilde{\mu}_n(m)$, satisfy

$$\lim_{n\to\infty} \frac{\tilde{\mu}_n(2m)}{\tilde{\mu}_n(2)^m} \;=\; (2m-1)!! \quad \text{and} \quad \lim_{n\to\infty} \frac{\tilde{\mu}_n(2m+1)}{\tilde{\mu}_n(2)^{m+\frac{1}{2}}} \;=\; 0. \tag{1.9}$$

Whereas Miller and Wang use generating functions to directly compute the moments $\tilde{\mu}_n(m)$, we instead compute them recursively (see, for example, Theorem 3.7).

For the rest of the paper, we omit standard algebra. The reader interested in the details of the omitted algebra should see [21] *for the details.*

1.3. **Organization of Paper.** In Section 2, we collect some notation and prior results to be used throughout the paper. In Section 3, we prove Theorem 3.3, the key technical lemma of the paper. In Section 4, Theorem 3.3 is used to prove Theorems 1.9, 1.10, and 1.11. In Section 5, we apply Theorem 3.3 to give alternate proofs of Theorems 1.6, 1.7, and 1.8, and conclude in Section 6 with a discussion of future work and open questions.

2. Preliminaries

We first collect some notation used throughout the paper, then isolate two technical lemmas on convergence, and then apply these to prove Gaussian behavior for certain two-dimensional recurrences. This final result is the basis for the proof of our main result on Gaussian behavior of gaps for a fixed g, Theorem 1.11.

2.1. **Notation.** For this paper, all big-Os are taken as $n \to \infty$, unless otherwise specified.

For a polynomial $A(x) = \sum_{k=0}^{d} a_k x^k$, let $\left[x^k\right](A(x)) = a_k$ be the notation for extracting the kth coefficient of A.

For a real number $\lambda_1 > 0$, a polynomial $A(x)$ has the *maximum root property with maximum root* λ_1 if λ_1 is a root of A with multiplicity 1 and all other roots have magnitude strictly less than λ_1.

A sequence of real numbers $\{a_n\}$ *converges exponentially quickly to* a if $\lim_{n\to\infty} a_n = a$ and there exists $\gamma \in (0,1)$ and a constant C such that $|a - a_n| \le C\gamma^n$ for all n (alternatively, $a_n = a + O(\gamma^n)$).

Let d be a fixed positive integer, and let $\{A_n(x)\}$ be a sequence of degree d polynomials, where $A_n(x) = \sum_{j=0}^{d} a_{j,n} x^j$. We say $\{A_n(x)\}$ *converges exponentially quickly to* $\bar{A}(x) = \sum_{j=0}^{d} \bar{a}_j x^j$ if $\{a_{j,n}\}_{n\in\mathbb{N}}$ converges exponentially quickly to $\bar{a}_j$ for $j = 0, 1, \dots, d$.

From the above definitions, we immediately obtain the following useful result.

Lemma 2.1. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences that converge exponentially quickly to* a *and* b*, respectively. Then,*

1. $\{a_n + b_n\}$ *converges exponentially quickly to* $a + b$*,*
2. $\{a_n - b_n\}$ *converges exponentially quickly to* $a - b$*,*
3. $\{a_n \cdot b_n\}$ *converges exponentially quickly to* $a \cdot b$*,*
4. *if* $b_n \neq 0$ *for all* n *and* $b \neq 0$*, then* $\{a_n / b_n\}$ *converges exponentially quickly to* a/b*.*

2.2. **Characteristic Polynomials.** Appendix A of [2] provides the following useful results.

Theorem 2.2 (Generalized Binet's Formula)**.** *Consider any linear recurrence of real numbers (not necessarily a positive linear recurrence)*

$$G_n \;=\; c_1 G_{n-1} + \cdots + c_L G_{n-L} \tag{2.1}$$

with arbitrary initial conditions. Suppose the characteristic polynomial $x^L - (c_1 x^{L-1} + c_2 x^{L-2} + \cdots + c_L)$ *has the maximum root property with some maximum root* $\lambda_1 > 0$*. Then, there exists a constant* a_1 *such that* $G_n = a_1 \lambda_1^n + O(n^{L-2}\lambda_2^n)$*, where* $|\lambda_2| < \lambda_1$ *is the second largest root in absolute value. Additionally, if* a_1 *is positive (that is,* $G_n = \Theta(\lambda_1^n)$*), then for every fixed positive integer* i*,* G_{n-i}/G_n *converges to* $1/\lambda_1^i$ *exponentially quickly as* $n \to \infty$*.*

Theorem 2.3. *Consider a PLRS* $\{G_n\}$ *given by*

$$G_n \;=\; c_1 G_{n-1} + \cdots + c_L G_{n-L}. \tag{2.2}$$

Then, the characteristic polynomial $x^L - (c_1 x^{L-1} + c_2 x^{L-2} + \cdots + c_L)$ *has the maximum root property with maximum root* $\lambda_1 > 1$ *and* $G_n = \Theta(\lambda_1^n)$*. In other words, the coefficient* a_1 *given by Theorem 2.2 is positive.*

Note in Theorem 2.2, a_1 is positive except for particular choices of initial conditions. For example, if $G_n = 5G_{n-1} - 6G_{n-2}$, we have $G_n = a_1 3^n + O(2^n)$, unless we have initial conditions $G_1 = \alpha$ and $G_2 = 2\alpha$, in which case the 3^n term vanishes.

3. Gaussian Behavior of 2D Recursions

The purpose of this section is to prove Theorem 3.3, the key ingredient to our main results. We start with a few technical lemmas.

3.1. **Convergence on Nonhomogenous Linear Recurrences with Noise.** The following two lemmas follow immediately from the previous definitions and book-keeping, and play a key role in the convergence analysis later. In particular, these two lemmas allow us to determine the exact behavior of the moments of our random variables $K_{g,n}$ as we prove convergence to the standard normal (see Lemmas 3.8 and 3.9).

Lemma 3.1. *Let* i_0 *be a positive integer. Let* $\{r_n\}_{n\in\mathbb{N}}$ *be a sequence of real numbers and for each* $1 \le i \le i_0$*, let* $\{s_{i,n}\}_{n\in\mathbb{N}}$ *be a sequence of nonnegative real numbers such that* $\sum_{i=1}^{i_0} s_{i,n} = 1$ *for all* n*. With a slight abuse of notation, also suppose there exist constants* $\bar{r}$ *and* $\bar{s}_i$ *for* $1 \le i \le i_0$*, with* $\gamma_r, \gamma_s \in (0,1)$ *such that*

$$r_n \;=\; \bar{r} + O(\gamma_r^n) \text{ and } s_{i,n} \;=\; \bar{s}_i + O(\gamma_s^n). \tag{3.1}$$

Furthermore, suppose the polynomial

$$S(x) \;=\; x^{i_0} - \sum_{i=1}^{i_0} \bar{s}_i x^{i_0 - i} \tag{3.2}$$

has the maximum root property with maximum root 1. Let $\{a_n\}_{n\ge n_0}$ *be a sequence with arbitrary initial conditions* $a_{n_0}, \ldots, a_{n_0+i_0-1}$ *and, for* $n \ge n_0 + i_0$*,*

$$a_n \;=\; \left(\sum_{i=1}^{i_0} s_{i,n} a_{n-i}\right) + r_n. \tag{3.3}$$

Then, there exists a positive integer d and a real number $\gamma \in (0,1)$ such that

$$a_n \;=\; \frac{\bar{r}}{\sum_{i=1}^{i_0} i \cdot \bar{s}_i} \cdot n + d + O(\gamma^n). \tag{3.4}$$

Roughly speaking, Lemma 3.1 is true because, modulo exponentially small terms, every a_n is a constant plus the weighted average of previous a_{n-i}'s, so it should be linear in n.

Proof. It suffices to prove the lemma for $n_0 = 0$. Let $b_n = a_n - \frac{\bar{r}}{\sum_{i=1}^{i_0} i \cdot \bar{s}_i} \cdot n$. Set $\gamma = \max(\gamma_r, \gamma_s)$. Simple manipulations yield

$$\begin{aligned} b_n \;&=\; \sum_{i=1}^{i_0} s_{i,n} b_{n-i} + \bar{r} \cdot \left(\frac{r_n}{\bar{r}} - \frac{\sum_{i=1}^{i_0} i \cdot \bar{s}_{i,n}}{\sum_{i=1}^{i_0} i \cdot \bar{s}_i} \right) \\ &=\; \sum_{i=1}^{i_0} s_{i,n} b_{n-i} + \bar{r} \cdot ((1 + O(\gamma^n)) - (1 + O(\gamma^n))) \\ &=\; \sum_{i=1}^{i_0} s_{i,n} b_{n-i} + O(\gamma^n). \end{aligned} \tag{3.5}$$

We finish by showing that the sequence b_n converges exponentially quickly to a constant. Simple algebra yields that b_n is bounded (see Appendix A of [21]). Then,

$$\begin{aligned} b_n - \sum_{i=1}^{i_0} \bar{s}_i b_{n-i} \;&=\; b_n - \sum_{i=1}^{i_0} s_{i,n} b_{n-i} + \sum_{i=1}^{i_0} (s_{i,n} - \bar{s}_i) b_{n-i} \\ &\le\; O(\gamma^n) + \sum_{i=1}^{i_0} O(\gamma^n) \cdot b_{n-i} \;\le\; O(\gamma^n). \end{aligned} \tag{3.6}$$

Thus, we can write

$$b_n \;=\; \left(\sum_{i=1}^{i_0} \bar{s}_i b_{n-i} \right) + f(n) \tag{3.7}$$

for some function $f : \{i_0, i_0 + 1, \dots\} \to \mathbb{R}$ such that $f(n) = O(\gamma^n)$ as $n \to \infty$. Let $\alpha_f > 0$ be a constant such that $|f(n)| \le \alpha_f \gamma^n$.

From here, the intuition to the finish is as follows. If $f(n) = 0$ for all n, then Theorem 2.2 implies that b_n approaches a constant exponentially quickly. However, since $\gamma < 1$, we have that b_n should still approach a constant exponentially quickly when $f(n) = O(\gamma^n)$.

Let $\{b_n^{(\mathrm{init})}\}_{n \in \mathbb{N}}, \{b_n^{(i_0)}\}_{n \in \mathbb{N}}, \{b_n^{(i_0+1)}\}_{n \in \mathbb{N}}, \dots$ be sequences defined (for $m \ge i_0$) by

$$\begin{aligned} b_n^{(\mathrm{init})} \;&=\; \begin{cases} b_n & 0 \le n \le i_0 - 1 \\ \sum_{i=1}^{i_0} \bar{s}_i b_{n-i}^{(\mathrm{init})} & n > i_0, \end{cases} \\ b_n^{(m)} \;&=\; \begin{cases} 0 & n < m \\ f(m) & n = m \\ \sum_{i=1}^{i_0} \bar{s}_i b_{n-i}^{(m)} & n > m. \end{cases} \end{aligned} \tag{3.8}$$

By induction, we verify that

$$b_n \;=\; b_n^{(\mathrm{init})} + \sum_{m=i_0}^{\infty} b_n^{(m)} \tag{3.9}$$

for all n (see Appendix B of [21]). By the restrictions of s_i, the characteristic polynomials of $\{b_n^{(\text{init})}\}$ and $\{b_n^{(m)}\}$ are equal to $S(x)$ in (3.2) and thus, have the maximum root property with maximum root 1. Hence, by the generalized Binet's formula, $\{b_n^{(\text{init})}\}$ and $\{b_n^{(m)}\}$ all converge to a constant. Suppose that $\{b_n^{(\text{init})}\}$ converges to $\bar{b}^{(\text{init})}$ and $\{b_n^{(m)}\}$ converges to $\bar{b}^{(m)}$ for each $m \geq i_0$. Let $\lambda_2 < 1$ be the second largest magnitude of a root of $S(x)$. Choose $\gamma_1 \in (\max(\gamma, \lambda_2), 1)$. By the generalized Binet's formula,

$$b_n^{(\text{init})} - \bar{b}^{(\text{init})} \;=\; O(n^{i_0} \cdot \lambda_2^n) \;\leq\; O(\gamma_1^n), \tag{3.10}$$

so there exists $\alpha_b^{(1)}$ such that

$$\left| b_n^{(\text{init})} - \bar{b}^{(\text{init})} \right| \;\leq\; \alpha_b^{(1)} \gamma_1^n. \tag{3.11}$$

For all m, we can bound $b_n^{(m)}$ similarly. However, note that for all m, $\{b_n^{(m)}/f(m)\}_{n\in\mathbb{N}}$ is the same sequence with the indices shifted. Thus, there exists $\alpha_b^{(2)}$ such that

$$\left| b_n^{(m)} - \bar{b}^{(m)} \right| \;\leq\; \alpha_b^{(2)} f(m) \gamma_1^{n-m} \;\leq\; \alpha_b^{(2)} \alpha_f \gamma^m \gamma_1^{n-m}. \tag{3.12}$$

Set $\alpha_b = \max(\alpha_b^{(1)}, \alpha_b^{(2)})$. Then,

$$\begin{aligned} |b_n - b| \;&\leq\; \left| b^{(\text{init})} - b_n^{(\text{init})} \right| + \sum_{m=i_0}^{\infty} \left| b^{(m)} - b_n^{(m)} \right| \\ &\leq\; \alpha_b \gamma_1^n + \sum_{m=i_0}^{\infty} \alpha_b \alpha_f \gamma_1^n \left(\frac{\gamma}{\gamma_1} \right)^m \\ &\leq\; \gamma_1^n \left(\alpha_b + \alpha_b \alpha_f \cdot \left(\frac{\gamma}{\gamma_1} \right)^{i_0} \cdot \frac{1}{1 - \frac{\gamma}{\gamma_1}} \right) \;=\; O(\gamma_1^n) \end{aligned} \tag{3.13}$$

as desired. □

The next lemma generalizes Lemma 3.1.

Lemma 3.2. *Let D be a nonnegative integer and let i_0 be a positive integer. Let $\{R_n(x)\}_{n\in\mathbb{N}}$ be a sequence of D degree polynomials with $R_n(x) = \sum_{j=0}^{D} r_{j,n} x^j$. For each $1 \leq i \leq i_0$, let $\{s_{i,n}\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that $\sum_{i=1}^{i_0} s_{i,n} = 1$ for all n. Suppose also that there exist a polynomial $\bar{R}(x) = \bar{r}_D x^D + \bar{r}_{D-1} x^{D-1} + \cdots + \bar{r}_0$ and real numbers $\bar{s}_i$ for $1 \leq i \leq i_0$, with $\gamma_r, \gamma_s \in (0,1)$, such that, for all $0 \leq j \leq D$ and $1 \leq i \leq i_0$,*

$$r_{j,n} \;=\; \bar{r}_j + O(\gamma_r^n), \text{ and } s_{i,n} \;=\; \bar{s}_i + O(\gamma_s^n). \tag{3.14}$$

Furthermore, suppose the polynomial, $S(x) = x^{i_0} - \sum_{i=1}^{i_0} \bar{s}_i x^{i_0 - i}$ has the maximum root property with maximum root 1. Let $\{a_n\}_{n \geq n_0}$ be a sequence with arbitrary initial conditions $a_{n_0}, \ldots, a_{n_0+i_0-1}$, and for $n \geq n_0 + i_0$,

$$a_n \;=\; \left(\sum_{i=1}^{i_0} s_{i,n} a_{n-i} \right) + R_n(n). \tag{3.15}$$

Then, there exists a degree $D+1$ polynomial $Q(x)$ and a $\gamma_1 \in (0,1)$ such that

$$a_n \;=\; Q(n) + O(\gamma_1^n), \tag{3.16}$$

where

$$\left[x^{D+1}\right](Q(x)) \;=\; \frac{\bar{r}_D}{(D+1)\sum_{i=1}^{i_0} i\cdot \bar{s}_i}. \tag{3.17}$$

In contrast to Lemma 3.1, a_n is, modulo exponentially small terms, a D degree polynomial in n plus the weighted average of previous a_{n-i}'s. Because for any D degree polynomial $A(x)$, the sum $A(1)+A(2)+\cdots+A(n)$ is an $D+1$ degree polynomial in n, we expect a_n to essentially be a $D+1$ degree polynomial in n.

Proof. We proceed by induction on D, the degree of the polynomials $R_n(x)$. Lemma 3.1 establishes the base case $D=0$. Now, assume $D>0$ and that the assertion is true for $D-1$. Let $b_n = a_n - C\cdot n^{D+1}$ for $C = \frac{\bar{r}_D}{(D+1)\sum_{i=1}^{i_0} i\cdot\bar{s}_i}$. Straightforward manipulations yield

$$b_n \;=\; \sum_{i=1}^{i_0} s_{i,n}b_{n-i} + \sum_{j=0}^{D-1} n^j\cdot\left[\left(\sum_{i=1}^{i_0} Cs_{i,n}(-1)^{D+1-j}\binom{D+1}{j} i^{D+1-j}\right) + r_{j,n}\right] + f(n) \tag{3.18}$$

for some function $f(n)\le O(\gamma_0^n)$ for some $\gamma_0\in(0,1)$ (see Appendix C of [21]). The constant C is chosen so that the right side contains an $D-1$ degree polynomial in n, as opposed to a D degree polynomial, which is the case in the recursion for $\{a_n\}$. Let $R_n^*(x) = \sum_{j=0}^{D-1} r_{j,n}^* x^j$ be the polynomial given by

$$\begin{aligned} r_{0,n}^* \;&:=\; \left(\sum_{i=1}^{i_0} Cs_{i,n}(-1)^{D+1}\binom{D+1}{0} i^{D+1}\right) + r_{0,n} + f(n) \\ \text{and } r_{j,n}^* \;&:=\; \left(\sum_{i=1}^{i_0} Cs_{i,n}(-1)^{D+1-j}\binom{D+1}{j} i^{D+1-j}\right) + r_{j,n} \end{aligned} \tag{3.19}$$

for $1\le j\le D-1$. Because, as $n\to\infty$, $s_{i,n}$ converges exponentially quickly to $\bar{s}_i$, $r_{j,n}$ converges exponentially quickly to $\bar{r}_j$, and $f(n)$ converges exponentially quickly to 0, we have $r_{j,n}^*$ converges exponentially quickly to

$$\lim_{n\to\infty} r_{j,n}^* \;=\; \left(\sum_{i=1}^{i_0} C\bar{s}_i(-1)^{D+1-j}\binom{D+1}{j} i^{D+1-j}\right) + \bar{r}_j \tag{3.20}$$

for $0\le j\le D-1$ by Lemma 2.1. Writing

$$b_n \;=\; \left(\sum_{i=1}^{i_0} s_{i,n}b_{n-i}\right) + R_n^*(n), \tag{3.21}$$

we can apply the induction hypothesis to b_n to obtain a degree D polynomial $Q^*(x)$ such that $b_n = Q^*(n) + O(\gamma_1^n)$ for some $\gamma_1\in(0,1)$. Set $Q(x) = Q^*(x) + Cx^{D+1}$. Then, $Q(x)$ is a degree $D+1$ polynomial satisfying $a_n = Q(x) + O(\gamma_1^n)$, as desired. □

3.2. **Gaussian Behavior of 2D Recursions.** The result below is the key ingredient in proving Gaussian behavior of gaps.

Theorem 3.3. *Let i_0 and j_0 be positive integers. Let $t_{i,j}$ be real numbers for $1\le i\le i_0$ and $0\le j\le j_0$ such that for all i, $\hat{t}_i := \sum_{j=0}^{j_0} t_{i,j}\ge 0$. Suppose that the polynomial $T(x) =$*

$x^{i_0} - \sum_{i=1}^{i_0} \hat{t}_i x^{i_0-i}$ *has the maximum root property with some maximum root* λ_1. *Suppose* $p_{n,k}$ *is a two-dimensional recurrence sequence satisfying, for* $n \geq n_0$,

$$p_{n,k} \;=\; \sum_{i=1}^{i_0}\sum_{j=0}^{j_0} t_{i,j} p_{n-i,k-j}. \tag{3.22}$$

Furthermore, suppose $p_{n,k} \geq 0$ *for all* n *and* k, $p_{n,k} = 0$ *when* $n < 0$ *or* $k < 0$, *finitely many* $p_{n,k}$ *are nonzero for* $n < n_0$, *and* $\sum_{i=0}^{\infty} p_{n,i} = \Theta(\lambda_1^n)$. *Let* X_n *be the random variable whose mass function is proportional to* $p_{n,k}$ *over varying* k *so that*

$$\mathbf{Pr}[X_n = k] \;=\; \frac{p_{n,k}}{\sum_{i=0}^{\infty} p_{n,i}}. \tag{3.23}$$

Let

$$C_\mu \;=\; \frac{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} \frac{t_{i,j}\cdot j}{\lambda_1^i}}{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} \frac{t_{i,j}\cdot i}{\lambda_1^i}} \text{ and } C_\sigma \;=\; \frac{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} \frac{t_{i,j}}{\lambda_1^i}\cdot (j - C_\mu i)^2}{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} \frac{t_{i,j}\cdot i}{\lambda_1^i}} \tag{3.24}$$

be constants, and assume $C_\sigma > 0$. *Then, there exist constants* $d_\mu, d_\sigma \in \mathbb{R}$, *and* $\gamma_\mu, \gamma_\sigma \in (0,1)$ *such that* $\mu_n = C_\mu n + d_\mu + O(\gamma_\mu^n)$ *and* $\sigma_n^2 = C_\sigma n + d_\sigma + O(\gamma_\sigma^n)$. *Furthermore,* $(X_n - \mu_n)/\sigma_n$ *converges weakly to the standard normal* $N(0,1)$ *as* $n \to \infty$.

In this theorem, imagine we have fixed a gap size g and think of $p_{n,k}$ as the number of $M \in [G_n, G_{n+1})$ whose decomposition has exactly k gaps of size g. Under this interpretation, the random variable X_n is identical to $K_{g,n}$. Note that $T(x)$ having the maximum root property does not make the condition $\sum_{i=0}^{\infty} p_{n,i} = \Theta(\lambda_1^n)$ redundant for reasons illustrated at the end of Section 2.1. This condition is necessary in Corollary 3.6.

We approach this problem using the method of moments, a common method for proving random variables converge in distribution to the standard normal distribution.

Lemma 3.4 (Method of Moments)**.** *Suppose* $X_1, X_2, \ldots$ *are random variables such that, for all integers* $m \geq 0$ *we have*

$$\lim_{n\to\infty} \mathbf{E}[X_n^{2m}] \;=\; (2m-1)!! \quad \textit{and} \quad \lim_{n\to\infty} \mathbf{E}[X_n^{2m+1}] \;=\; 0. \tag{3.25}$$

Then, the sequence $X_1, X_2, \ldots$ *converges weakly in distribution to the standard normal* $N(0,1)$.

The proof of Theorem 3.3 proceeds by using generating functions to compute the moments of X_n. Let

$$\begin{aligned}
P_n(x) &= \sum_{k=0}^{\infty} p_{n,k} x^k, \\
\Omega_n &= P_n(1) = \sum_{k=0}^{\infty} p_{n,k}, \\
\tilde{P}_{n,0}(x) &= \frac{P_n(x)}{x^{\mu+1}}, \\
\tilde{P}_{n,m}(x) &= (x\tilde{P}_{n,m-1}(x))', \\
\tilde{\mu}_n(m) &= \frac{\tilde{P}_{n,m}(1)}{\Omega_n}.
\end{aligned} \tag{3.26}$$

Then, it follows from definitions that

$$\begin{aligned} \mu_n &= \frac{P_n'(1)}{P_n(1)}, \\ \tilde{\mu}_n(m) &= \mathbf{E}[(X_n - \mu_n)^m], \\ \sigma_n^2 &= \tilde{\mu}_n(2). \end{aligned} \tag{3.27}$$

Now, we prove several lemmas about the above moments and generating functions. We ultimately obtain a formula in Theorem 3.7 that recursively computes the moments $\tilde{\mu}_n(m)$, which yield Theorem 3.3.

The next lemma follows immediately from the definitions (see [21] for details).

Lemma 3.5. *For $n \geq n_0$, we have*

$$P_n(x) = \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} t_{i,j} P_{n-i}(x) x^j. \tag{3.28}$$

From the above, we immediately deduce the following relations.

Corollary 3.6. *For $n \geq n_0$, we have*

$$\Omega_n = P_n(1) = \tilde{P}_{n,0}(1) = \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} t_{i,j} P_{n-i}(1) = \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} t_{i,j} \Omega_{n-i} \tag{3.29}$$

and

$$\mu_n = \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} \frac{t_{i,j}\Omega_{n-i}}{\Omega_n} (\mu_{n-i} + j). \tag{3.30}$$

By definition of Ω_n, we have $\Omega_n = \Theta(\lambda_1^n)$, so by Theorem 2.2, we have for all i, Ω_{n-i}/Ω_n converges exponentially quickly to $1/\lambda_1^i$.

Theorem 3.7. *For $n \geq n_0$, we have*

$$\tilde{\mu}_n(m) = \sum_{\ell=0}^{m} \binom{m}{\ell} \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} \frac{\Omega_{n-i} t_{i,j}}{\Omega_n} \cdot (j + \mu_{n-i} - \mu_n)^{\ell} \cdot \tilde{\mu}_{n-i}(m - \ell). \tag{3.31}$$

Proof. Applying Lemma 3.5, we find

$$\tilde{P}_{n,0}(x) = \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} t_{i,j} \tilde{P}_{n-i,0}(x) \cdot x^{j + \mu_{n-i} - \mu_n}. \tag{3.32}$$

By induction, we can establish (see Appendix D of [21])

$$\tilde{P}_{n,m}(x) = \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} t_{i,j} \sum_{\ell=0}^{m} \binom{m}{\ell} (j + \mu_{n-i} - \mu_n)^{\ell} \tilde{P}_{n-i,m-\ell}(x) \cdot x^{j + \mu_{n-i} - \mu_n} \tag{3.33}$$

so,

$$\begin{aligned} \tilde{\mu}_n(m) = \frac{\tilde{P}_{n,m}(1)}{\Omega_n} &= \frac{1}{\Omega_n} \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} t_{i,j} \sum_{\ell=0}^{m} \binom{m}{\ell} (j + \mu_{n-i} - \mu_n)^{\ell} \tilde{P}_{n-i,m-\ell}(1) \\ &= \sum_{\ell=0}^{m} \binom{m}{\ell} \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} \frac{\Omega_{n-i} t_{i,j}}{\Omega_n} \cdot (j + \mu_{n-i} - \mu_n)^{\ell} \cdot \tilde{\mu}_{n-i}(m - \ell), \end{aligned} \tag{3.34}$$

completing the proof. □

Our next goal is to prove

$$\lim_{n\to\infty} \frac{\tilde{\mu}_n(2m)}{\tilde{\mu}_n(2)^m} \ = \ (2m-1)!! \quad \text{and} \quad \lim_{n\to\infty} \frac{\tilde{\mu}_n(2m+1)}{\tilde{\mu}_n(2)^{m+\frac{1}{2}}} \ = \ 0. \tag{3.35}$$

By Lemma 3.4, these equalities imply Theorem 3.3. To prove these equalities, we first show μ_n is essentially linear in n. Then, we determine for all m, the behavior of $\tilde{\mu}_n(m)$, the mth moment of $X_n - \mu_n$, up to an exponentially small term. We prove $\tilde{\mu}_n(m)$ is a degree (at most, if m is odd) $\lfloor m/2 \rfloor$ polynomial in n, and for even moments $\tilde{\mu}_n(2m)$, we additionally compute the leading coefficient of this polynomial. We rely heavily on Lemmas 3.1 and 3.2 to determine the polynomial behavior of the moments.

Lemma 3.8. *There exists a real number d_μ and a $\gamma_\mu \in (0,1)$ such that*

$$\mu_n \ = \ C_\mu \cdot n + d_\mu + O(\gamma_\mu^n). \tag{3.36}$$

Proof. Recall

$$C_\mu \ = \ \frac{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} \frac{t_{i,j}\cdot j}{\lambda_1^i}}{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} \frac{t_{i,j}\cdot i}{\lambda_1^i}}. \tag{3.37}$$

Choose $s_{i,n} = \frac{\Omega_{n-i}}{\Omega_n}\sum_{j=0}^{j_0} t_{i,j} = \frac{\Omega_{n-i}}{\Omega_n}\hat{t}_i$ and $r_n = \sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot j\cdot \Omega_{n-i}}{\Omega_n}$. Using Lemma 2.1 and Corollary 3.6, we have, for each i, $s_{i,n}$ converges exponentially quickly to $\bar{s}_i = \frac{1}{\lambda_1^i}\sum_{j=0}^{j_0} t_{i,j} = \hat{t}_i$ and r_n converges exponentially quickly to $\bar{r} = \sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot j}{\lambda_1^i}$. By Corollary 3.6, we have

$$\mu_n \ = \ \left(\sum_{i=1}^{i_0} s_{i,n}\mu_{n-i}\right) + r_n. \tag{3.38}$$

Furthermore, the polynomial $S(x) = x^{i_0} - \sum_{i=1}^{i_0} \bar{s}_i x^{i_0-i}$ satisfies $S(x) = T(x/\lambda_1)$, so S has the maximum root property with maximum root 1. Then, by Lemma 3.1, there exist $d_\mu \in \mathbb{R}$ and $\gamma_\mu \in (0,1)$ such that

$$\mu_n \ = \ \frac{\bar{r}}{\sum_{i=1}^{i_0} i\bar{s}_i}\cdot n + d_\mu + O(\gamma_\mu^n) \ = \ C_\mu \cdot n + d_\mu + O(\gamma_\mu^n). \tag{3.39}$$

□

Lemma 3.9. *For each integer $m \geq 0$, there exist $\gamma_{2m}, \gamma_{2m+1} \in (0,1)$ and polynomials Q_{2m} of degree exactly m and Q_{2m+1} of degree at most m such that*

$$\begin{aligned}\tilde{\mu}_n(2m) \ &= \ Q_{2m}(n) + O(\gamma_{2m}^n)\\ \text{and } \tilde{\mu}_n(2m+1) \ &= \ Q_{2m+1}(n) + O(\gamma_{2m+1}^n).\end{aligned} \tag{3.40}$$

Furthermore, if $C_{2m} = [x^m]Q_{2m}$ and $C_{2m+1} = [x^m]Q_{2m+1}$, then for all $m \geq 0$, $C_{2m} = (2m-1)!!\cdot C_\sigma^m$ (We take $(-1)!! = 1$).

The idea for the proof is as follows. In the calculation of $\mu_n(m)$ in Theorem 3.7, the coefficients of $\mu_{n-i}(m)$ sum to 1, the coefficients of $\mu_{n-i}(m-1)$ sum to 0, and the coefficients of $\mu_{n-i}(m-2)$ sum to $\binom{m}{2}\cdot(\text{constant})$. Thus, the mth moments can be written in the form of (3.15), so we can apply Lemma 3.2 and compute the degrees and leading coefficients appropriately. Because the coefficients of the $(m-1)$th moments sum to 0, the degrees of the polynomials increase by one with every two values of m, as opposed to every one.

Proof. We proceed by induction on m. The base case $m = 0$ follows from noting that

$$\begin{aligned} \tilde{\mu}_n(0) &= \mathbf{E}[(X_n - \mu_n)^0] = 1 \\ \text{and } \tilde{\mu}_n(1) &= \mathbf{E}[(X_n - \mu_n)^1] = 0 \end{aligned} \tag{3.41}$$

for all $n \geq n_0$. Now, assume the statement is true for $m' \leq m$. That is, there exist $\gamma_0, \gamma_1, \ldots, \gamma_{2m-1} \in (0, 1)$ and polynomials $Q_0, Q_1, \ldots, Q_{2m-1}$, where Q_k has degree $k/2$ when k is even, and degree at most $\lfloor k/2 \rfloor$ when k is odd, such that

$$\begin{aligned} \tilde{\mu}_n(2m-2) &= Q_{2m-2}(n) + O(\gamma_{2m-2}^n) \\ \text{and } \tilde{\mu}_n(2m-1) &= Q_{2m-1}(n) + O(\gamma_{2m-1}^n). \end{aligned} \tag{3.42}$$

By induction, we may further assume $C_{2m-2} = (2m-3)!! \cdot C_\sigma^{m-1}$. First, we compute $\tilde{\mu}_n(2m)$. Define a sequence of polynomials $\{R_n(x)\}$ via

$$R_n(x) = \sum_{\ell=1}^{2m} \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} \frac{\Omega_{n-i} t_{i,j}}{\Omega_n} \cdot (j + \mu_{n-i} - \mu_n)^\ell \cdot Q_{2m-\ell}(x - i). \tag{3.43}$$

Furthermore, set

$$s_{i,n} = \frac{\Omega_{n-i}}{\Omega_n} \sum_{j=0}^{j_0} t_{i,j} \quad \text{and} \quad \bar{s}_i = \frac{1}{\lambda_1^i} \sum_{j=0}^{j_0} t_{i,j}. \tag{3.44}$$

Then,

$$\tilde{\mu}_n(2m) = \sum_{i=1}^{i_0} s_{i,n} \tilde{\mu}_{n-i}(2m) + R_n(n). \tag{3.45}$$

Note $R_n(x)$ is the sum of finitely many polynomials that, by Lemma 2.1, converges exponentially quickly. Thus, $R_n(x)$ converges exponentially quickly to

$$\bar{R}(x) = \sum_{\ell=1}^{2m} \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} \frac{t_{i,j}}{\lambda_1^i} \cdot (j - C_\mu i)^\ell \cdot Q_{2m-\ell}(x - i). \tag{3.46}$$

Furthermore, we have $\deg \bar{R}(x) \leq m - 1$ because each $R_n(x)$ has degree at most $m - 1$. With (3.46), we can compute $[x^{m-1}]\bar{R}(x)$. Because $Q_{2m-\ell}$ has degree at most $m - 2$ for $\ell \geq 3$ and the coefficients in front of the Q_{2m-1} terms sum to 0, by the definition of C_μ, we have (see [21] for details)

$$\begin{aligned} [x^{m-1}](\bar{R}(x)) &= \sum_{\ell=1}^{2m} \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} \binom{2m}{\ell} \frac{t_{i,j}}{\lambda_1^i} \cdot (j - C_\mu i)^\ell \cdot [x^{m-1}](Q_{2m-\ell}(x - i)) \\ &= C_{2m-2} \cdot \binom{2m}{2} \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} \frac{t_{i,j}}{\lambda_1^i} \cdot (j - C_\mu i)^2 \\ &= C_{2m-2} \cdot \binom{2m}{2} \cdot C_\sigma \cdot \left(\sum_{i=1}^{i_0} i \cdot \bar{s}_i \right). \end{aligned} \tag{3.47}$$

By Lemma 3.2, there exists a degree $\deg \bar{R}(x) + 1$ polynomial $Q_{2m}(x)$ with x^m coefficient C_{2m} and a $\gamma_{2m} \in (0, 1)$ such that

$$\mu_n(2m) = Q_{2m}(n) + O(\gamma_{2m}^n) \tag{3.48}$$

and

$$C_{2m} \;=\; \frac{C_{2m-2}\cdot\binom{2m}{2}\cdot C_\sigma\cdot\left(\sum_{i=1}^{i_0} i\cdot\bar{s}_i\right)}{m\cdot\sum_{i=1}^{i_0} i\cdot\bar{s}_i} \;=\; C_{2m-2}\cdot(2m-1)\cdot C_\sigma. \tag{3.49}$$

By the inductive hypothesis, we conclude $C_{2m} = (2m-1)!!\cdot C_\sigma^m$. By our technical assumption, $C_\sigma \neq 0$, so $C_{2m}\neq 0$ and thus, the degree of Q_{2m} is exactly m.

We can perform the same computation to show that the $\tilde{\mu}_n(2m+1)$ can be expressed as the sum of an mth degree polynomial in n and an exponentially small term. To see this, define a sequence of polynomials $\{R_n(x)\}$ via

$$R_n(x) \;=\; \sum_{\ell=1}^{2m+1}\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{\Omega_{n-i}t_{i,j}}{\Omega_n}\cdot(j+\mu_{n-i}-\mu_n)^\ell\cdot Q_{2m+1-\ell}(x-i). \tag{3.50}$$

Just as in the $2m$th moments case, set

$$s_{i,n} \;=\; \frac{\Omega_{n-i}}{\Omega_n}\sum_{j=0}^{j_0} t_{i,j}. \tag{3.51}$$

Then,

$$\tilde{\mu}_n(2m+1) \;=\; \sum_{i=1}^{i_0} s_{i,n}\tilde{\mu}_{n-i}(2m+1) + R_n(n). \tag{3.52}$$

Note that $R_n(x)$ is the sum of finitely many polynomials that, by Lemma 2.1, converge exponentially quickly. Thus, $R_n(x)$ converges exponentially quickly to

$$\bar{R}(x) \;=\; \sum_{\ell=1}^{2m+1}\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}}{\lambda_1^i}\cdot(j-C_\mu i)^\ell\cdot Q_{2m+1-\ell}(x-i). \tag{3.53}$$

Furthermore, we have $\deg\bar{R}(x)\leq m-1$. Indeed, Q_{2m} has degree m, so to show that $\deg\bar{R}(x)\leq m-1$, we need to show that the coefficient of x^m is 0. Looking at the x^m coefficients of (3.53) gives

$$\begin{aligned}[x^m](\bar{R}(x)) \;&=\; \sum_{\ell=1}^{2m}\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\binom{2m}{\ell}\frac{t_{i,j}}{\lambda_1^i}\cdot(j-C_\mu i)^\ell\cdot[x^m](Q_{2m+1-\ell}(x-i))\\ &=\; C_{2m}\cdot\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}2m\cdot\frac{t_{i,j}}{\lambda_1^i}\cdot(j-C_\mu i)^1 \;=\; C_{2m}\cdot 2m\cdot 0 \;=\; 0.\end{aligned} \tag{3.54}$$

The second to last equality follows from the definition of C_μ in (3.24). Again, applying Lemma 3.2 gives that there exists a degree $\deg\bar{R}(x)+1$ polynomial $Q_{2m+1}(x)$ such that $\tilde{\mu}_n(2m+1) = Q_{2m+1}(n)+O(\gamma_{2m+1}^n)$. Because $\deg\bar{R}(x)+1\leq m$, this completes the induction. □

Proof of Theorem 3.3. Lemma 3.8 proves the first part of Theorem 3.3. Lemma 3.9 implies that $\sigma_n^2 = \tilde{\mu}_n(2) = Q_2(n)+O(\gamma_2^n)$. Writing $Q_2(n) = C_\sigma n + d_\sigma$ for some $d_\sigma\in\mathbb{R}$, we have $\sigma_n^2 = C_\sigma n + d_\sigma + O(\gamma_2^n)$, proving the second part of Theorem 3.3. We finish the proof of Theorem 3.3 by noting (3.35) is an immediate consequence of Lemma 3.9. □

4. Gap Theorems

4.1. **Gap Recurrence.** We start by finding a recurrence relation for an $M \in [G_n, G_{n+1})$ having exactly k gaps of size g. Recall that $k_g(M)$ denotes the number of gaps of size g in the Zeckendorf decomposition of M.

Lemma 4.1. *Let $\{G_n\}$ be a positive linear recurrence with recurrence relation*

$$G_n \ = \ c_1 G_{n-1} + \cdots + c_L G_{n-L} \tag{4.1}$$

and $c_i > 0$ for all i. Slightly abusing the notation (reusing the letter p), let

$$p_{g,n,k} \ = \ |\{M \in [G_n, G_{n+1}) : k_g(M) \ = \ k\}|. \tag{4.2}$$

Define $d_0 = 0$ and $d_i = c_1 + c_2 + \cdots + c_i$ for $1 \le i \le L$ and set $c_i^* = c_i$ for $1 \le i < L$ and $c_L^* = c_L - 1$. Then, there exists $n_0 = L + g$ and $k_0 = d_L$ such that, for $n \ge n_0$, $k \ge k_0$, and $g \ge 2$, we have

$$\begin{aligned}
p_{0,n,k} \ &= \ \sum_{i=1}^{L} \sum_{j=1}^{c_i - 1} p_{0,n-i,k-(d_{i-1}-(i-1)+(j-1))} + \sum_{i=1}^{L} p_{0,n-i,k-(d_{i-1}-(i-1))} \\
p_{1,n,k} \ &= \ p_{1,n-1,k} + \sum_{i=1}^{L} (c_i - 1) p_{1,n-i,k-(i-1)} + \sum_{i=2}^{L} p_{1,n-i,k-(i-2)} \\
&\quad + \sum_{i=1}^{L} (c_i - 1) \left(\left(p_{1,n-i,k-i} - p_{1,n-i,k-(i-1)} \right) - \left(p_{1,n-i-1,k-i} - p_{1,n-i-1,k-(i-1)} \right) \right) \\
p_{g,n,k} \ &= \ \sum_{i=1}^{L} c_i p_{g,n-i,k} + \sum_{i=1}^{L} c_i^* \left((p_{g,n+1-i-g,k-1} - p_{g,n+1-i-g,k}) - (p_{g,n-i-g,k-1} - p_{g,n-i-g,k}) \right).
\end{aligned} \tag{4.3}$$

Proof. Define

$$q_{g,n,k} \ = \ |\{M \in [1, G_n) : k_g(M) = k\}| \ = \ \sum_{i=1}^{n-1} p_{g,i,k}; \tag{4.4}$$

thus, whereas $p_{g,n,k}$ is the number of M in $[G_n, G_{n+1})$ such that $k_g(M) = k$, $q_{g,n,k}$ is the corresponding quantity for integers in $[1, G_n)$. Set $H_{n,0} = 0$ and $H_{n,i} = \sum_{i'=1}^{i} c_{i'} G_{n+1-i'}$ so that, for all n, $H_{n,L} = G_{n+1}$. Let

$$\Gamma \ = \ \{(i,j) \in \mathbb{Z}^2 : 0 \le i \le L-1, 0 \le j \le c_{i+1} - 1, (i,j) \ne (0,0)\}. \tag{4.5}$$

For $n \in \mathbb{N}$ and $(i,j) \in \Gamma$, let $I_{n,i,j} = [H_{n,i} + jG_{n-i}, H_{n,i} + (j+1)G_{n-i})$ be an interval of integers. The $c_1 + c_2 + \cdots + c_L - 1$ intervals $\{I_{n,i,j} : (i,j) \in \Gamma\}$ form a partition of $[G_n, G_{n+1})$, and the sequential order of these intervals is equal to their lexicographical order by (i,j). For each $(i,j) \in \Gamma$, we can express $|\{M \in I_{n,i,j} : k_g(M) = k\}|$ in terms of $p_{g,n,k}$ and $q_{g,n,k}$ with smaller values of n. This is done by case work on whether the smallest term in $H_{n,i} + jG_{n-i}$

(either G_{n+1-i} or G_{n-i} depending on whether $j = 0$) is part of a gap of size g:

$$\begin{aligned}
|\{M \in I_{n,i,0} : k_0(M) = k\}| &= q_{0,n-i,k-(d_i-i)} \\
|\{M \in I_{n,i,0} : k_1(M) = k\}| &= q_{1,n-i,k-(i-1)} \\
|\{M \in I_{n,i,0} : k_g(M) = k\}| &= q_{g,n-i,k} + p_{g,n+1-i-g,k-1} - p_{g,n+1-i-g,k} \\
|\{M \in I_{n,i,j} : k_0(M) = k\}| &= q_{0,n-i,k-(d_i-i+(j-1))} \\
|\{M \in I_{n,i,j} : k_1(M) = k\}| &= q_{1,n-i,k-i} + p_{g,n-i-1,k-(i+1)} - p_{g,n-i-1,k-i} \\
|\{M \in I_{n,i,j} : k_g(M) = k\}| &= q_{g,n-i,k} + p_{g,n-i-g,k-1} - p_{g,n-i-g,k} \qquad (4.6)
\end{aligned}$$

(see Appendix E.1 of [21] for details). These formulas are clean because the number of size g gaps in an $M = H_{n,i} + jG_{n-i} + M' \in I_{n,i,j}$ is the number of size g gaps in $H_{n,i} + jG_{n-i}$ plus the number of size g gaps in M' plus possibly one more gap between the two decompositions. By definition, for $g \geq 0$ we have

$$p_{g,n,k} = \sum_{(i,j)\in\Gamma} |\{M \in I_{n,i,j} : k_g(M) = k\}|. \qquad (4.7)$$

From this equation, we can substitute from (4.6), plug in the results for $p_{g,n,k}$ and $p_{g,n-1,k}$ in the expression $p_{g,n,k} - p_{g,n-1,k}$, use the identity $q_{g,n,k} - q_{g,n-1,k} = p_{g,n-1,k}$, and apply straightforward manipulations to obtain the desired result (see Appendix E.2 of [21] for calculations). □

4.2. **Proof of Gap Theorems.** Lemma 4.1 allows us to apply Theorem 3.3 to the distribution of the number of fixed sized gaps. The proof is essentially verifying that the conditions of Theorem 3.3 are met by our gap recurrences.

Proofs of Theorems 1.9, 1.10, and 1.11. Recall that $k_g(M)$ denotes the number of gaps of size g in the Zeckendorf decomposition of M. Let

$$p_{g,n,k} = |\{M \in [G_n, G_{n+1}) : k_g(M) = k\}| \qquad (4.8)$$

and let $i_0 = L+g$ and $j_0 = d_L$. By Lemma 4.1, for every $g \geq 0$, there exist $t_{i,j}$ for $1 \leq i \leq L+g$ and $0 \leq j \leq d_L$ such that, for $n > i_0$,

$$p_{g,n,k} = \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} t_{i,j} p_{g,n-i,k-j}. \qquad (4.9)$$

Define $\hat{t}_i = \sum_{j=0}^{j_0} t_{i,j}$. Note that in each recursive formula of (4.3), the terms of the form $p_{g,n-x,y_1} - p_{g,n-x,y_2}$ contribute 0 to $\sum_{j=0}^{j_0} t_{x,j}$, and for each $0 \leq i \leq L-1$, the remaining coefficients of $p_{g,n-i-1,k}$ (over varying k) sum to c_{i+1}. From this, we conclude $\hat{t}_i = c_i$ for $1 \leq i \leq L$ and $\hat{t}_i = 0$ for $L < i \leq i_0$. Thus, the polynomial

$$T(x) = x^{i_0} - \sum_{i=1}^{i_0} \hat{t}_i x^{i_0-i} = x^{i_0-L}\left(x^L - \sum_{i=1}^{L} c_i x^{L-i}\right) \qquad (4.10)$$

has the maximum root property with some maximum root $\lambda_1 > 1$, by Theorem 2.3. Also, $\sum_{k=0}^{n} p_{g,n,k} = G_{n+1} - G_n = \Theta(\lambda_1^n)$ by Theorems 2.2 and 2.3. Because $p_{g,n,k}$ counts something that is well-defined when $n \geq 1$ and $k \geq 0$, we have $p_{g,n,k} \geq 0$ for all n and k; and $p_{g,n,k} = 0$ for $n < 0$ or $k < 0$. Also, $p_{g,n,k} = 0$ for all $k \geq n$, because no $M \in [G_n, G_{n+1})$ can have a gap greater than n. Thus, there are finitely many pairs (n,k) with $n \le i_0$ such that $p_{g,n,k} \neq 0$. Lastly, for every g, if the random variable $K_{g,n}$ is nontrivial, then the $t_{i,j}$ satisfy

$$C_\mu \;=\; \frac{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot j}{\lambda_1^i}}{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot i}{\lambda_1^i}} \text{ and } C_\sigma \;=\; \frac{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}}{\lambda_1^i}\cdot(j-C_\mu i)^2}{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot i}{\lambda_1^i}}. \tag{4.11}$$

To prove $C_\mu > 0$ and $C_\sigma > 0$, we split this into cases whether or not $g = 0$, $g = 1$, or $g \ge 2$. For each case we substitute into (4.11) and perform standard manipulations (see Appendix F of [21]). Putting these observations together, the proofs follow by applying Theorem 3.3. □

5. Lekkerkerker and Gaussian Summands

We show the power of Theorem 3.3 by reproving Theorems 1.6, 1.7, and 1.8. We borrow from the proof, given by Miller and Wang [22], of the recursion established for $p_{n,k}$, the number of $M \in [G_n, G_{n+1})$ with exactly k summands. This recursion is extracted as (5.1) from generating functions in [22]. This recursion can also be found using techniques in the proof of Lemma 4.1; the casework for number of summands is simpler than for gaps. Miller and Wang's arguments show the mean and variance grow linearly in n, but many technical calculations are needed to show the linear coefficients are positive (which is a key ingredient in the proof of the Gaussian behavior). See [7] for another approach, which bypasses the difficulties through an elementary argument involving conditional probabilities, or [3] for a proof using Markov processes.

Similar to Section 4.2, the proof is essentially verifying that the conditions of Theorem 3.3 are met by the summands recursion given by Miller and Wang.

Proofs of Theorems 1.6, 1.7, and 1.8. Let $p_{n,k}$ be the number of $M \in [G_n, G_{n+1})$ with exactly k summands. Then $\mathbf{Pr}[K_{\Sigma,n} = k] = \frac{p_{n,k}}{\sum_{k=0}^{\infty} p_{n,k}}$. Again, $p_{n,k} \ge 0$ for all n and k and $p_{n,k} = 0$ for all $n < 0$ and $k < 0$. Also, $p_{n,k} > 0$ for finitely many pairs with $n < L$ as $p_{n,k} = 0$ for all $k > n \cdot \max_i(c_i)$ and because each M has, for each $a \in \{1,\dots,n\}$, at most $\max_i(c_i)$ copies of G_a in each decomposition.

Define $d_i = c_1 + c_2 + \cdots + c_i$ for $1 \le i \le L$. By Proposition 3.1 from [23], $p_{n,k}$ satisfies, for $n \ge L$ and $k \ge d_L$,

$$p_{n,k} \;=\; \sum_{i=1}^{L}\sum_{j=d_{m-1}}^{d_m-1} p_{n-i,k-j}. \tag{5.1}$$

For $1 \le i \le L$ and $0 \le j < d_L$, set $t_{i,j}$ to be 1 if $d_{i-1} \le j < d_i - 1$ and 0 otherwise. Defining $\hat{t}_i = \sum_{j=0}^{d_L-1} t_{i,j}$ gives $\hat{t}_i = c_i$, and the polynomial

$$T(x) \;=\; x^{i_0} - \sum_{i=1}^{i_0}\hat{t}_i x^{i_0-i} \;=\; x^{i_0-L}\left(x^L - \sum_{i=1}^{L} c_i x^{L-i}\right) \tag{5.2}$$

has the maximum root property with some maximum root $\lambda_1 > 1$ by Theorem 2.3. Also, $\sum_{k=0}^{n} p_{g,n,k} = G_{n+1} - G_n = \Theta(\lambda_1^n)$ by Theorems 2.2 and 2.3. Lastly, because all the $t_{i,j}$ are nonnegative and $t_{n-L,k-(d_L-1)} = 1$ with $k-(d_L-1) > 0$, (3.24) tells us

$$C_\mu \;=\; \frac{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot j}{\lambda_1^i}}{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot i}{\lambda_1^i}} \;\ge\; \frac{\frac{k-(d_L-1)}{\lambda_1^{n-L}}}{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0}\frac{t_{i,j}\cdot i}{\lambda_1^i}} \;>\; 0. \tag{5.3}$$

Because $t_{1,0} = 1$ and all the $t_{i,j}$ are nonnegative, we have

$$C_\sigma \;=\; \frac{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} \frac{t_{i,j}}{\lambda_1^i}\cdot (j - C_\mu i)^2}{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} \frac{t_{i,j}\cdot i}{\lambda_1^i}} \;\geq\; \frac{\frac{t_{1,0}}{\lambda_1^1}\cdot (0 - C_\mu 1)^2}{\sum_{i=1}^{i_0}\sum_{j=0}^{j_0} \frac{t_{i,j}\cdot i}{\lambda_1^i}} \;>\; 0. \tag{5.4}$$

Thus, we can apply Theorem 3.3, implying the theorems. □

6. Further Work and Open Questions

We end with a few natural questions for future work.

(1) Are there other two-dimensional recurrences to which we can apply our Central Limit type result? The second author is investigating two-dimensional sequences and associated notions of legality with colleagues. These lead to recurrence relations, although the resulting sequences do not have unique decomposition.
(2) Can one remove the constraint that every coefficient c_i must be positive and obtain the same results? Notice that with negative constraints, one loses some of the interpretations for the algebra.
(3) What is the rate at which $K_{g,n}$ converges to a normal distribution?

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