DIOPHANTINE EQUATIONS WITH THE RAMANUJAN τ FUNCTION OF FACTORIALS, FIBONACCI NUMBERS, AND CATALAN NUMBERS

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Abstract. In this note, we solve various Diophantine equations of the type $|\tau(x)| = y$, where τ is the Ramanujan τ function and x and y are integer variables restricted to values of factorials, Fibonacci numbers, and Catalan numbers.

1. Introduction

Various Diophantine equations of the form $f(x) = y$ have been solved in the literature. Here, f is some function defined on the set of positive integers, like the Euler function or the sum of divisors function. And x and y are integer variables restricted to various interesting subsets of the positive integers, like the factorials, the Fibonacci numbers, the Catalan numbers, or other members of interesting sequences of integers. See, for example, [2, 3, 4, 5]. In this paper, the function f is the Ramanujan function $\tau(n)$, defined as the coefficient of q^n of the following series,

$$q\left(\prod_{k\geq 1}(1-q^k)\right)^{24} = \sum_{n\geq 1}\tau(n)q^n \qquad \text{for} \qquad |q|<1.$$

Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$ and let C_n be the nth Catalan number given by

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

In this note, we prove the following theorem.

Theorem 1.1.

(1) *The only positive integer solutions* (m, n) *of the Diophantine equation*

$$|\tau(m!)| = F_n$$

are $(1, 1)$ *and* $(1, 2)$.

(2) *The only positive integer solution* (m, n) *of the Diophantine equation*

$$|\tau(m!)| = C_n$$

is $(1, 1)$.

(3) *The only positive integer solutions* (m, n) *of the Diophantine equation*

$$|\tau(C_m)| = F_n$$

are $(1, 1)$ *and* $(1, 2)$.

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(4) *The only positive integer solution* (m,n) *of the Diophantine equation*

$$|\tau(C_m)| = C_n.$$

is $(1,1)$.

2. Auxiliary Results

Putting $(\alpha,\beta) = ((1+\sqrt{5})/2, (1-\sqrt{5})/2)$, we have that

$$F_k = \frac{\alpha^k - \beta^k}{\alpha - \beta} \quad \text{for all} \quad k \ge 1.$$

In particular,

$$\alpha^{k-2} \le F_k \le \alpha^{k-1} \quad \text{holds for all} \quad k \ge 1. \tag{2.1}$$

Concerning the Ramanujan function, it is well-known that $|\tau(m)| < 2m^6$ for all $m \ge 1$ (see page 182 in [1]). In particular,

$$|\tau(m!)| < 2m!^6 \le 2(m^{m-1})^6 = \left(\frac{2}{m^m}\right) m^{6m} < m^{6m} \quad \text{for all} \quad m \ge 2. \tag{2.2}$$

Finally, concerning C_m, we use

$$2^m < C_m < \frac{2^{2m}}{m+1} \quad \text{for all} \quad m \ge 3. \tag{2.3}$$

For a prime number p and a positive integer m, let $\nu_p(m)$ denote the exponent of p in the factorization of m. For a positive real number x, we use $\pi(x)$ for the number of primes $p \le x$.

Lemma 2.1. *The following properties hold:*

(i) $\nu_2(m!) \ge m/2$ *for all* $m \ge 4$.
(ii) $\nu_2(\tau(2^a)) \ge 3a$ *for all* $a \ge 0$.
(iii) $\nu_2(\tau(m!)) \ge 3m/2$ *for all* $m \ge 2$.
(iv) $\nu_2(F_n) \le \log(4n/3)/\log 2$ *for all* $n \ge 1$.
(v) $\nu_2(C_m) \le \log(2m)/\log 2$ *for all* $m \ge 1$.
(vi) $\pi(2m) - \pi(m+1) \ge m/(2\log m)$ *for all* $m \ge 19$.

Proof. Part (i) follows from

$$\nu_2(m!) = \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{4} \right\rfloor + \cdots \ge \left\lfloor \frac{m}{2} \right\rfloor + 1 > \frac{m}{2}$$

for $m \ge 4$. Part (ii) follows by induction on a via $\tau(1) = 1$, $\tau(2) = -24$, and

$$\tau(2^{a+2}) = -24\tau(2^{a+1}) - 2048 \quad \text{for all} \quad a \ge 0.$$

Part (iii) follows immediately from (i) and (ii) for $m \ge 4$ and one checks that it also holds for $m = 2, 3$ because $\tau(2) = -2^3 \cdot 3$ and $\tau(6) = -2^5 \cdot 3^3 \cdot 7$. For part (iv) one checks, by induction on k, that for $k \ge 3$, we have that $2^k \mid F_n$ if and only if $3 \times 2^{k-2} \mid n$. Thus, if $k = \nu_2(F_n)$ and $k \ge 3$, then $3 \times 2^{k-2} \le n$, so $k \le \log(4n/3)/\log 2$. One checks that the inequality (iv) also holds for $k < 3$ by noting that k can never be 2, whereas for $k = 1$, inequality (iv) follows because in this case $3 \mid n$, so $n \ge 3$. Part (v) follows because

$$\nu_2(C_m) = \left(\left\lfloor \frac{2n}{2} \right\rfloor - 2\left\lfloor \frac{n}{2} \right\rfloor\right) + \left(\left\lfloor \frac{2n}{4} \right\rfloor - 2\left\lfloor \frac{n}{4} \right\rfloor\right) + \cdots + \left(\left\lfloor \frac{2n}{2^k} \right\rfloor - 2\left\lfloor \frac{n}{2^k} \right\rfloor\right) + \cdots.$$

Furthermore, each of the above parenthesis is in $\{0,1\}$ and the largest k for which the parenthesis is nonzero satisfies $2^k \le 2n$, so $k \le \log(2n)/\log 2$. For part (v), recall from [7] that the following estimates hold:

$$\frac{x}{\log x - 0.5} < \pi(x) < \frac{x}{\log x - 1.5} \quad \text{for all} \quad x \ge 67.$$

In particular,

$$\pi(2m) - \pi(m+1) > \frac{2m}{\log(2m) - 0.5} - \frac{m+1}{\log(m+1) - 1.5}.$$

The right side exceeds $m/(2\log m)$ for all $m \ge 160$. A quick computation shows that $\pi(2m) - \pi(m+1) > m/(2\log m)$ for all $m \in [19, 160]$. □

3. The Proofs

First, we show $n < 1000$. Assume $n \ge 1000$. Clearly, $m > 1$. Then,

$$\alpha^{n-2} < F_n = |\tau(m!)| < m^{6m}$$

by (2.1) and (2.2). Suppose $m < n/(18\log n)$. We then get

$$\alpha^{n-2} < m^{6m} < \exp(6m\log m) < \exp(n/3),$$

which gives

$$(3\log\alpha)(n-2) < n, \qquad \text{so} \qquad n < \frac{6\log\alpha}{3\log\alpha - 1} < 7,$$

a contradiction. Thus, $m \ge n/(18\log n)$. Since $n > 438$, it follows that

$$m > \frac{n}{18\log n} > 4,$$

therefore,

$$\nu_2(m!) > \frac{m}{2} > \frac{n}{36\log n} \tag{3.1}$$

by Lemma 2.1 (i). Thus,

$$\frac{\log(4n/3)}{\log 2} \ge \nu_2(F_n) = \nu_2(\tau(m!)) \ge 3\nu_2(m!) > \frac{n}{12\log n}$$

by Lemma 2.1 (i), (iii), and (iv). This gives $n \le 809$, contradicting that $n \ge 1000$. Thus, $n \le 1000$. It then follows that

$$\nu_2(F_n) \le \frac{\log(4n/3)}{\log 2} \le 10.4,$$

so $\nu_2(|\tau(m!)|) \le 10$, which shows that $\nu_2(m!) \le 3$. Thus, $m \in \{1,2,3,4,5\}$, so

$$F_n \in \{1, 24, 6048, 21288960, 102825676800\},$$

and the only Fibonacci number in the above set is $1 = F_1 = F_2$.

For part (ii), assume $n > 1000$. Note that

$$\alpha^{n-2} < \alpha^n < 2^n < C_n = |\tau(m!)| < m^{6m},$$

and now the previous argument shows that $m \ge n/(18\log n)$. We thus get that

$$\frac{\log(2n)}{\log 2} \ge \nu_2(C_n) = \nu_2(\tau(m!)) > \frac{n}{12\log n}$$

by Lemma 2.1 (i), (iii), and (v). This gives $n \le 876$, contradicting that $n > 1000$. Thus, $n \le 1000$. It then follows that

$$\nu_2(C_n) \le \frac{\log(2n)}{\log 2} \le 10.97,$$

so $\nu_2(|\tau(m!)|) \le 10$, which shows that $\nu_2(m!) \le 3$. Thus, $m \in \{1,2,3,4,5\}$, so

$$C_n \in \{1, 24, 6048, 21288960, 102825676800\},$$

and the only Catalan number in the above set is $1 = C_1$.

For (iii), we use (2.3) to deduce that

$$|\tau(C_m)| < 2C_m^6 < 2\left(\frac{2^{2m}}{m+1}\right)^6 < 2^{12m-2} \quad \text{for} \quad m \ge 1.$$

Thus,

$$2^{12m-2} > F_n > \alpha^{n-2}, \quad \text{so} \quad 2^{12m} > \left(\frac{4}{\alpha^2}\right)\alpha^n > \alpha^n$$

giving

$$m > \frac{(\log \alpha)n}{12 \log 2} > \frac{n}{18}.$$

By Lemma 2.1 (vi), for $m \ge 19$, the interval $(m+1, 2m)$ contains at least $m/(2\log m)$ primes. For each one of such primes p, we have that $p \| C_m$ and $\tau(p)$ is even. Hence, by multiplicativity

$$\frac{\log(4n/3)}{\log 2} \ge \nu_2(F_n) = \nu_2(\tau(C_m)) \ge \pi(2m) - \pi(m+1) \ge \frac{m}{2\log m} > \frac{n}{36 \log n},$$

so $n \le 4000$. This shows that

$$\nu_2(\tau(C_m)) \le \nu_2(F_n) \le \frac{\log(4n/3)}{\log 2} < 12.4,$$

so $\nu_2(\tau(C_m)) \le 12$. Thus, there are at most 12 primes in the interval $(m+1, 2m)$, which gives $m \le 63$. We now list all values of $|\tau(C_m)|$ for $m \in [1, 63]$ and all values of F_n for $n \in [1, 4000]$ and intersect these two sets and get the conclusion.

The same argument applies to (iv). Namely,

$$2^{12m-2} > |\tau(C_m)| = C_n > 2^n, \quad \text{therefore,} \quad m > \frac{n}{12}.$$

We then get that

$$\frac{\log(2n)}{\log 2} \ge \nu_2(C_n) = \nu_2(\tau(m!)) \ge \frac{m}{2\log m} > \frac{n}{24 \log n},$$

giving $n \le 2500$. Thus,

$$\nu_2(\tau(C_m)) \le \frac{\log(2n)}{\log 2} \le 12.3,$$

so again $\nu_2(\tau(C_m)) \le 12$, which implies that $m \le 63$. We now list all values of $|\tau(C_m)|$ for $m \in [1, 63]$ and all values of C_n for $n \in [1, 2500]$ and intersect these two sets and get the conclusion.

4. Conclusion

We did not get any interesting solutions for the equations studied in our main result. Allowing $|\tau(m!)|$ to be a product of more Fibonacci numbers, we got the following solutions:

$$\begin{aligned} |\tau(1!)| &= F_1; \\ |\tau(2!)| &= F_4F_6; \\ |\tau(3!)| &= F_3^5F_4^2F_8; \\ |\tau(4!)| &= F_3^{11}F_4^2F_8F_{10}; \\ |\tau(5!)| &= F_3^7F_4F_5F_8F_{10}F_{24}. \end{aligned}$$

We suggest the following problem.

Problem 4.1. *Find all solutions of the equation*

$$|\tau(m!)| = F_{n_1}F_{n_2}\cdots F_{n_k}$$

in positive integers $m,\ n_1 \le n_2 \le \cdots \le n_k$.

Without the Ramanujan function, the largest solution of the equation $m! = F_{n_1}\cdots F_{n_k}$ is $11! = F_1F_2F_3F_4F_5F_6F_8F_{10}F_{12}$ (see [6]).

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