ARITHMETIC FUNCTIONS OF FIBONACCI AND LUCAS NUMBERS

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Abstract. Let F_n and L_n be the nth Fibonacci and Lucas numbers, respectively. Let $\varphi(n)$ be the Euler totient function of n and $\sigma_k(n)$ the sum of kth powers of the positive divisors of n. Luca obtained the inequalities $\varphi(F_n) \geq F_{\varphi(n)}$, $\sigma_0(F_n) \geq F_{\sigma_0(n)}$, and $\sigma_k(F_n) \leq F_{\sigma_k(n)}$ for all $n, k \geq 1$. In this article, we extend Luca's result by replacing the function φ by φ_k and J_k, which are generalizations of φ. We also consider the corresponding results for $\varphi_k(L_n)$, $L_{\varphi_k(n)}$, $J_k(L_n)$, $L_{J_k(n)}$, $\sigma_k(L_n)$, and $L_{\sigma_k(n)}$.

1. Introduction

Throughout this article, p is a prime, k is a nonnegative integer, m and n are positive integers, and F_n and L_n are the nth Fibonacci and Lucas numbers, respectively. Here $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$. In addition, let $\sigma_k(n)$ be the sum of kth powers of the positive divisors of n, $\varphi(n)$ the number of elements in a reduced residue system modulo n, or more generally,

$$\varphi_k(n) = \sum_{\substack{1 \leq m \leq n \\ (m,n)=1}} m^k \quad \text{and} \quad J_k(n) = n^k \prod_{p \mid n} \left(1 - \frac{1}{p^k}\right). \tag{1.1}$$

Therefore, $\varphi_0(n) = \varphi(n) = J_1(n)$. The divisibility property of F_n and the behavior of $\sigma_k(n)$, $\varphi_k(n)$, $J_k(n)$, and values of other number-theoretic functions have been a popular area of research. For some recent results on this topic, we refer the reader to [12, 13, 15, 21, 22] and references therein. In particular, Luca [9] showed that $\varphi(F_n) \geq F_{\varphi(n)}$, $\sigma_0(F_n) \geq F_{\sigma_0(n)}$, and $\sigma_k(F_n) \leq F_{\sigma_k(n)}$ for all $n, k \geq 1$, which was extended to the case of balancing numbers by Sahukar and Panda [24]. Luca and Young [11] claimed that $\sigma_0(F_n) \mid F_n$ if and only if $n \in \{1, 2, 3, 6, 24, 48\}$. Bugeaud, Luca, Mignotte, and Siksek [2] gave a description of F_n for which $\omega(F_n) \leq 2$, and Pongsriiam extended the results on $\omega(F_n)$ further in [18]. Here, $\omega(F_n)$ is the number of distinct prime factors of F_n. For other problems involving arithmetic functions or Fibonacci numbers, see for example in Broughan, et al. [1], Luca and Shparlinski [10], and Pongsriiam [17, 16].

In this article, we extend Luca's result [9] by replacing the function φ by its generalizations φ_k and J_k. We also consider the corresponding results for $\varphi_k(L_n)$, $L_{\varphi_k(n)}$, $J_k(L_n)$, $L_{J_k(n)}$, $\sigma_k(L_n)$, and $L_{\sigma_k(n)}$. We organize this article as follows. In Section 2, we prove some auxiliary results for the reader's convenience. In Section 3, we show the inequalities between $g(F_n)$, $F_{g(n)}$, $g(L_n)$, and $L_{g(n)}$, where $g = \varphi_k$, J_k, or σ_k. Then we give some open problems at the end of Section 3.

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2. Preliminaries and Lemmas

Suppose $n \in \mathbb{N}$ and p is a prime. Recall that the p-adic valuation of n, denoted by $\nu_p(n)$, is the exponent of p in the prime factorization of n. The order (or the rank) of appearance of n in the Fibonacci sequence, denoted by $z(n)$, is the smallest positive integer k such that $n \mid F_k$. The results concerning Fibonacci and Lucas numbers, which are needed in the proof of the main theorems, are as follows.

Lemma 2.1. *Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then, the following statements hold.*

(i) *(Binet's Formula) $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$ for all $n \geq 1$.*
(ii) *$F_{2n} = F_n L_n$ and $L_{2n} = {L_n}^2 - 2(-1)^n$ for all $n \geq 1$.*
(iii) *$F_n > n$ for all $n \geq 6$.*
(iv) *$F_{2^n} > 13^n$ for all $n \geq 5$.*
(v) *$\alpha^{n-2} \leq F_n \leq \alpha^{n-1}$ for all $n \geq 1$.*
(vi) *$\alpha^{n-1} \leq L_n \leq \alpha^{n+1}$ for all $n \geq 1$.*
(vii) *If p is a prime and $p \neq 5$, then $z(p) \mid p - (p \mid 5)$, where $(p \mid 5)$ is the Legendre symbol. In particular, $z(p) \leq p + 1$.*
(viii) *$5 \nmid L_n$ for all $n \geq 1$.*
(ix) *$2 \mid L_n$ if and only if $3 \mid n$.*

Proof. Statements (i), (ii), (viii), and (ix) are well-known, see for example in [7]. For (vii) and other properties of $z(n)$, we refer the reader to [3, 4, 5, 6, 14, 20, 25] and references therein. Statements (iii), (iv), (v), and (vi) can be proved by induction. Here, we only give a proof of (iv) because the other proofs are similar. We first check directly that $F_{32} > 13^5$. If $n \geq 5$ and $F_{2^n} > 13^n$, then we obtain by (ii) that $F_{2^{n+1}} = F_{2^n} L_{2^n} > (F_{2^n})^2 > 13^{2n} > 13^{n+1}$. This completes the proof. □

Lemma 2.2. (Lengyel [8]) *Suppose $n \in \mathbb{N}$ and p is a prime distinct from 2 and 5. Then,*

$$\nu_p(L_n) = \begin{cases} \nu_p(n) + \nu_p(F_{z(p)}), & \text{if } z(p) \text{ is even and } n \equiv \frac{z(p)}{2} \pmod{z(p)}; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.3. *The following statements hold.*

(i) *$\frac{L_{2n}}{L_{2n-1}} > \frac{L_{2n+2}}{L_{2n+1}} > \alpha$ for all $n \geq 1$.*
(ii) *$\frac{L_{2n+1}}{L_{2n}} < \frac{L_{2n+3}}{L_{2n+2}} < \alpha$ for all $n \geq 1$.*
(iii) *$\frac{L_{1+n^k}}{L_n^k} > \frac{29}{18}$ for all $n \geq 7$ and $k \geq 1$.*

Proof. By Lemma 2.1, to prove the first inequality in (i), it is enough to show that

$$(\alpha^{2n} + \beta^{2n})(\alpha^{2n+1} + \beta^{2n+1}) > (\alpha^{2n+2} + \beta^{2n+2})(\alpha^{2n-1} + \beta^{2n-1}). \tag{2.1}$$

The left side of (2.1) is equal to $\alpha^{4n+1} + \beta^{4n+1} + \alpha(\alpha\beta)^{2n} + \beta(\alpha\beta)^{2n} = L_{4n+1} + L_1$. Similarly, the right side of (2.1) is $L_{4n+1} - L_3$, which is less than $L_{4n+1} + L_1$. For the second inequality in (i), we have $\alpha L_{2n+1} = \alpha^{2n+2} + \alpha\beta^{2n+1} = \alpha^{2n+2} - \beta^{2n} < \alpha^{2n+2} + \beta^{2n+2} = L_{2n+2}$. This proves (i). The proof of (ii) is similar. Next, we prove (iii). Let $n \geq 7$ and $k = 1$. If $k = 1$, then by (i) and (ii), we obtain

$$\frac{L_{1+n^k}}{L_n^k} = \frac{L_{n+1}}{L_n} > \frac{L_7}{L_6} = \frac{29}{18}.$$

Suppose $k \geq 2$. Then by Lemma 2.1, we have $L_n^k \leq \alpha^{(n+1)k} < \alpha^{n^k-1} \leq L_{n^k}$. Therefore,

$$\frac{L_{1+n^k}}{L_n^k} > \frac{L_{1+n^k}}{L_{n^k}} > \frac{L_7}{L_6} = \frac{29}{18}.$$

This completes the proof. □

Next, we prove some inequalities involving φ_k and σ_k.

Lemma 2.4. *The following statements hold.*

(i) *For* $n \geq 10$, $\varphi(n) > \frac{n}{5\log\log n}$.
(ii) *For* $n < 2 \times 10^9$, $\frac{n}{\varphi(n)} < 16$.
(iii) *For* $n > 2 \times 10^9$, $\frac{n}{\varphi(n)} < \log n$.

Proof. In a straightforward way, we check that (i) holds for $10 \leq n \leq 19$. We also know from [23, (3.42)] that $\varphi(n) > \frac{n}{e^c \log\log n + \frac{2.50637}{\log\log n}}$, where $n \geq 3$ and $c = 0.5772\ldots$ is the Euler constant. This implies (i) for $n \geq 20$. Since $5\log\log(2 \times 10^9) < 16$, (ii) follows immediately from (i). By a more careful analysis, Ward [27, Lemmas 4.1 and 4.2] obtained $n/\varphi(n) < 6$. But, (ii) is good enough for our calculation. Inequality (iii) is also obtained by Ward [27]. Alternatively, we can prove (iii) by using (i) again. This completes the proof. □

Lemma 2.5. *Let* $n \geq 3$. *Then* $\varphi_1(n) = \frac{n\varphi(n)}{2}$ *and for* $k \geq 2$, *we have* $\frac{n^k\varphi(n)}{2^k} < \varphi_k(n) < \frac{n^k\varphi(n)}{2}$.

Proof. Observe that if m is a positive integer, $m < \frac{n}{2}$, and $(m,n) = 1$, then $n/2 < n - m < n$ and $(n - m, n) = 1$. Conversely, if $n/2 < m' < n$ and $(m', n) = 1$, then $m' = n - m$ for some m such that $1 \leq m < n/2$ and $(m,n) = 1$. Therefore, we can pair the integers m and $n - m$ in the sum defining $\varphi_k(n)$, with $\frac{\varphi(n)}{2}$ pairs, and write

$$\varphi_k(n) = \sum_{\substack{1\leq \ell \leq n\\(\ell,n)=1}} \ell^k = \sum_{\substack{1\leq m<n/2\\(m,n)=1}} \left(m^k + (n-m)^k\right). \tag{2.2}$$

We do not include $n/2$ in the sum because $n/2$ is not an integer or otherwise $(n/2, n) = n/2 > 1$. If $k = 1$, then (2.2) implies that $\varphi_1(n) = n\varphi(n)/2$. Suppose $k \geq 2$ and consider the function f defined by $f(x) = x^k + (n - x)^k$ for $0 \leq x \leq n$. By considering $f'(x)$ and recalling the well-known result in calculus, we see that f is strictly decreasing on $[0, n/2]$. So, $f(0) > f(m) > f(n/2)$ for all $m \in (0, n/2)$. Since there are $\varphi(n)/2$ pairs of $(m, n - m)$ in (2.2), we obtain

$$\varphi_k(n) = \sum_{\substack{1\leq m<n/2\\(m,n)=1}} f(m) < f(0)\frac{\varphi(n)}{2} = \frac{n^k\varphi(n)}{2}.$$

Similarly, $\varphi_k(n) > f(n/2)\varphi(n)/2 = \frac{n^k\varphi(n)}{2^k}$. This gives the desired result. □

Lemma 2.6. *Let* $m \geq 4$ *and* $k \geq 1$. *We have*

(i) $\frac{m}{\varphi(m)} > \frac{\sigma_k(m)}{m^k}$.
(ii) *If* m *is not prime, then* $\sigma_k(m) - m^k \geq 1 + \sqrt{m}^k$.

Proof. If p is a prime and $a \in \mathbb{N}$, then $\sigma_k(p^a)/p^{ak}$ is equal to

$$\frac{p^{ak} + p^{(a-1)k} + \ldots + p^k + 1}{p^{ak}} = \sum_{c=0}^{a} \frac{1}{p^{ck}} < \sum_{c=0}^{\infty} \frac{1}{p^c} = \left(1 - \frac{1}{p}\right)^{-1}.$$

If we write $m = {p_1}^{a_1} \cdots {p_\ell}^{a_\ell}$ in the canonical factorization and use the multiplicativity of $\sigma_k(m)/m^k$, then we obtain

$$\frac{\sigma_k(m)}{m^k} < \prod_{i=1}^{\ell} \left(1 - \frac{1}{p_i}\right)^{-1} = \frac{m}{\varphi(m)}.$$

This proves (i). Next, assume that m is not prime. Then there exists a divisor d of m such that $d \neq 1$, $d \neq m$, and $d \geq \sqrt{m}$. Since 1 and m are also divisors of m, we see that $\sigma_k(m) \geq 1 + m^k + d^k \geq 1 + m^k + \sqrt{m}^k$, which implies (ii). □

3. Main Results

Since $J_1(n) = \varphi(n) = \varphi_0(n)$ and Luca [9] already proved that $\varphi(F_n) \geq F_{\varphi(n)}$ for all $n \geq 1$, we check the inequalities between $J_k(F_n)$ and $F_{J_k(n)}$ only for $k \geq 2$. Similarly, we consider $\varphi_k(F_n)$ and $F_{\varphi_k(n)}$ only for $k \geq 1$. We begin with the following result.

Theorem 3.1. *Let $k \geq 2$. Then, the following statements hold.*

(i) *$J_k(F_n) \leq F_{J_k(n)}$ for all $n \geq 1$. In addition, $J_k(F_n) = F_{J_k(n)}$ if and only if $n = 1$.*
(ii) *$J_k(L_n) \leq L_{J_k(n)}$ for all $n \geq 1$ except when $n = 2$ and $k = 2$, where we have $J_2(L_2) > L_{J_2(2)}$. Furthermore, $J_k(L_n) = L_{J_k(n)}$ if and only if $n = 1$.*

Proof. We first verify (i) for $1 \leq n \leq 18$ by using Lemma 2.1 as follows. For $n \in \{1, 2, 3\}$, we have $J_k(F_1) = 1 = F_{J_k(1)}$, $F_{J_k(2)} = F_{2^k-1} \geq F_3 > J_k(F_2)$, and $F_{J_k(3)} = F_{3^k-1} > 3^k - 1 > 2^k - 1 = J_k(F_3)$. If $n = 4$, we check directly that $J_2(F_n) \leq F_{J_2(n)}$ and for $k \geq 3$, we have $F_{J_k(n)} = F_{4^k-2^k} > F_{2^{k+2}} > 13^{k+2} > 3^k - 1 = J_k(F_n)$. The case $5 \leq n \leq 18$ can be proved similarly, so we show the details only when $n = 8$. In a straightforward way, we check that $F_{J_k(8)} > J_k(F_8)$ for $k = 2, 3, 4$, and for $k \geq 5$, we have $F_{J_k(n)} = F_{8^k-4^k} > F_{2^{k+1}} = F_{2^k} L_{2^k} > 13^{2k} > 21^k > (3^k - 1)(7^k - 1) = J_k(F_8)$. Hence, (i) holds for $1 \leq n \leq 18$. Similarly, (ii) also holds for $1 \leq n \leq 18$. Therefore, we assume throughout that $n \geq 19$. Since $1 - \frac{1}{p^k} > 1 - \frac{1}{p} > \left(1 - \frac{1}{p}\right)^k$, we see that

$$J_k(n) = n^k \prod_{p \mid n} \left(1 - \frac{1}{p^k}\right) > \left(n \prod_{p \mid n} \left(1 - \frac{1}{p}\right)\right)^k = \varphi(n)^k.$$

Therefore, $F_{J_k(n)} > F_{\varphi(n)^k}$. In addition, $J_k(F_n) < {F_n}^k$, so it suffices to show that $F_{\varphi(n)^k} \geq {F_n}^k$. By Lemma 2.1, ${F_n}^k \leq \alpha^{(n-1)k}$ and $F_{\varphi(n)^k} \geq \alpha^{\varphi(n)^k - 2}$. It is enough to show that $\varphi(n)^k \geq (n-1)k + 2$. Similarly, to show that $L_{J_k(n)} \geq J_k(L_n)$, it is enough to show that $\varphi(n)^k \geq (n+1)k + 1$. Since $(n+1)k + 1 > (n-1)k + 2$, we only need to show that

$$\varphi(n)^k \geq (n+1)k + 1. \tag{3.1}$$

We first show that (3.1) holds for $19 \leq n \leq 135$ and $k \geq 2$ by induction on k. For $k = 2$, we ran a computation on a computer to see that $\varphi(n)^2 \geq 2n + 3$ for $19 \leq n \leq 135$. Suppose $k \geq 2$ and (3.1) holds for k. Then $\varphi(n)^{k+1} \geq \varphi(n)((n+1)k + 1) \geq 2(n+1)k + 2 \geq (n+1)(k+1) + 1$, as required. It remains to show that (3.1) holds for all $k \geq 2$ and $n \geq 136$. To apply Lemma 2.4, we first consider the function $f : [136, \infty) \to \mathbb{R}$ defined by

$$f(x) = \left(\frac{x}{5 \log \log x}\right)^k - k(x + 1) - 1.$$

Using calculus, we see that f is increasing on $[136,\infty)$ and therefore, $f(x) \geq f(136) > 17^k - 137k - 1 > 0$ for all $x \geq 136$. This implies $\left(\frac{x}{5\log\log x}\right)^k > k(x+1)+1$ for all $x \geq 136$. Then by Lemma 2.4, we obtain $\varphi(n)^k > \left(\frac{n}{5\log\log n}\right)^k > k(n+1)+1$ for all $n \geq 136$ and $k \geq 2$, as required. This completes the proof. □

Theorem 3.2. *Let $k \geq 1$. Then the following statements hold.*

(i) *$\varphi_k(F_n) \leq F_{\varphi_k(n)}$ for all $n \geq 1$ except when $n = 6$ and $k = 1$, where we have $\varphi_1(F_6) > F_{\varphi_1(6)}$. In addition, $\varphi_k(F_n) = F_{\varphi_k(n)}$ if and only if $n \in \{1,2\}$ or $(n,k) = (4,1)$.*

(ii) *$\varphi_k(L_n) \leq L_{\varphi_k(n)}$ for all $n \geq 1$ except when $n = 2$ or $(n,k) \in \{(4,1),(4,2),(6,1)\}$, where the inequality reverses. In addition, $\varphi_k(L_n) = L_{\varphi_k(n)}$ if and only if $n = 1$ or $(n,k) = (3,1)$.*

Proof. By Lemma 2.1 and a straightforward calculation, it is not difficult to verify (i) and (ii) for $1 \leq n \leq 12$. Assume throughout that $n \geq 13$. We first show that $\varphi(n) \geq 4$. If there exists a prime $p \geq 5$ dividing n, we obtain, by the multiplicativity of φ, that $\varphi(n) \geq \varphi(p) = p-1 \geq 4$. Suppose that $n = 2^a 3^b$ for some $a, b \in \mathbb{N} \cup \{0\}$. If $a \geq 3$, then $\varphi(n) \geq \varphi(2^3) = 4$. If $a = 2$, then $b \geq 1$ and $\varphi(n) \geq \varphi(4)\varphi(3) = 4$. If $a \leq 1$, then $b \geq 2$ and $\varphi(n) \geq \varphi(9) = 6$. In any case, $\varphi(n) \geq 4$, as desired.

Case 1: $k = 1$. By Lemmas 2.1 and 2.5, we obtain

$$F_{\varphi_k(n)} = F_{\frac{n\varphi(n)}{2}} \geq F_{2n} = F_n L_n > {F_n}^2 > F_n\varphi(F_n) > \frac{F_n\varphi(F_n)}{2} = \varphi_k(F_n),$$
$$L_{\varphi_k(n)} = L_{\frac{n\varphi(n)}{2}} \geq L_{2n} = {L_n}^2 - 2(-1)^n \geq {L_n}^2 - 2 > \frac{{L_n}^2}{2} > \frac{L_n\varphi(L_n)}{2} = \varphi_k(L_n).$$

Case 2: $k \geq 2$. By Lemma 2.5, $\varphi_k(F_n) \leq \frac{\varphi(F_n)}{2}{F_n}^k < \frac{{F_n}^{k+1}}{2} \leq \frac{\alpha^{(n-1)(k+1)}}{2} < \alpha^{(n-1)(k+1)-1}$. By Lemma 2.1, $F_{\varphi_k(n)} \geq \alpha^{\varphi_k(n)-2}$. So, it suffices to show that $\varphi_k(n) \geq (n-1)(k+1)+1$. Since $\varphi_k(n) \geq \varphi(n)\left(\frac{n}{2}\right)^k \geq 4\left(\frac{n}{2}\right)^k$, it is enough to show that $4\left(\frac{n}{2}\right)^k \geq (n-1)(k+1)+1$. Similarly, to show that $\varphi_k(L_n) \leq L_{\varphi_k(n)}$, it is enough to show that $4\left(\frac{n}{2}\right)^k \geq (n+1)(k+1)+1$. To prove (i) and (ii), we only need to show that

$$4\left(\frac{n}{2}\right)^k \geq (n+1)(k+1)+1. \tag{3.2}$$

We consider the function $f : [13,\infty) \to \mathbb{R}$ defined by

$$f(x) = 4\left(\frac{x}{2}\right)^k - (x+1)(k+1) - 1.$$

If $x \geq 13$, then $f'(x) = 2k(x/2)^{k-1} - (k+1) > 12k - (k+1) > 0$, and so f is increasing on $[13,\infty)$. Therefore, $f(x) \geq f(13) > 4(13/2)^k - 14(k+1) - 1 > 0$ for all $x \geq 13$. This implies $4\left(\frac{x}{2}\right)^k \geq (x+1)(k+1)+1$ for all $x \geq 13$. So, (3.2) holds and the proof is complete. □

Luca [9] obtained the inequality between $\sigma_k(F_n)$ and $F_{\sigma_k(n)}$. We make his result more complete by considering $\sigma_k(L_n)$ and $L_{\sigma_k(n)}$ in the next theorem, and then $\varphi(L_n)$ and $L_{\varphi(n)}$ in Theorem 3.4.

Theorem 3.3. *Let $k \geq 1$. Then $\sigma_k(L_n) \leq L_{\sigma_k(n)}$ for all $n \geq 1$ and $\sigma_k(L_n) = L_{\sigma_k(n)}$ if and only if $n = 1$ or $(n,k) \in \{(2,1),(3,1)\}$.*

Proof. In a straightforward manner, we check that $\sigma_k(L_n) = L_{\sigma_k(n)}$, when $n = 1$ and $k \in \mathbb{N}$ and when $(n,k) \in \{(2,1),(3,1)\}$. In addition, we also check that $L_{\sigma_k(2)} > \sigma_k(L_2)$ and $L_{\sigma_k(3)} > \sigma_k(L_3)$ for $k = 2,3,4$, and for $k \geq 5$, we have $L_{\sigma_k(2)} = L_{1+2^k} > F_{2^k} > 13^k > 3^k + 1 = \sigma_k(L_2)$ and $L_{\sigma_k(3)} = L_{1+3^k} > F_{2^k} > 13^k > 1 + 2^k + 4^k = \sigma_k(L_3)$. Similarly, $L_{\sigma_k(n)} > \sigma_k(n)$ for $n \in \{4,5\}$ and $k \geq 1$. Therefore, we assume that $k \geq 1$ and $n \geq 6$. We also ran a computation on a computer to verify that

$$\sigma_1(L_n) < L_{\sigma_1(n)} \text{ for } 6 \leq n \leq 110. \text{ So if } k = 1, \text{ we can assume that } n \geq 111. \tag{3.3}$$

Moreover, suppose for a contradiction that

$$\sigma_k(L_n) \geq L_{\sigma_k(n)}. \tag{3.4}$$

Next, we show that n must be a prime. Suppose n is not a prime.

Case 1: $n \leq 44$. By (3.3), we can assume $k \geq 2$. We have $L_n \leq L_{44} < 2 \cdot 10^9$. By Lemmas 2.4, 2.6, and (3.4), it follows that

$$L_6 = 18 > \frac{L_n}{\varphi(L_n)} > \frac{\sigma_k(L_n)}{{L_n}^k} \geq \frac{L_{\sigma_k(n)}}{{L_n}^k}. \tag{3.5}$$

Since $n^k \geq (n+1)k + 2$, we obtain, by Lemmas 2.1 and 2.6, that

$$\frac{L_{\sigma_k(n)}}{{L_n}^k} \geq \alpha^{\sigma_k(n)-1-(n+1)k} \geq \alpha^{\sigma_k(n)-n^k+1} \geq L_{\sigma_k(n)-n^k} \geq L_{\sqrt{n^k}} \geq L_n \geq L_6. \tag{3.6}$$

By (3.5) and (3.6), we obtain a contradiction.

Case 2: $n \geq 45$. Then $L_n \geq L_{45} > 2 \cdot 10^9$. It follows from Lemmas 2.4, 2.6, and (3.4) that

$$\log L_n > \frac{L_n}{\varphi(L_n)} > \frac{\sigma_k(L_n)}{{L_n}^k} \geq \frac{L_{\sigma_k(n)}}{{L_n}^k}. \tag{3.7}$$

If $k \geq 2$, then in a manner similar to (3.6), we have

$$\frac{L_{\sigma_k(n)}}{{L_n}^k} \geq L_{\sigma_k(n)-n^k} \geq \alpha^{\sigma_k(n)-n^k-1} \geq \alpha^{\sqrt{n^k}} \geq \alpha^n, \tag{3.8}$$

and then by (3.7) and (3.8) and using Lemma 2.1, we obtain

$$2(n+1) > \log \alpha^{n+1} \geq \log L_n > \alpha^n > 2(n+1),$$

which is a contradiction. Therefore, $k = 1$. By (3.3), we can assume $n \geq 111$. Then by Lemmas 2.1 and 2.6, $\frac{L_{\sigma_1(n)}}{L_n} \geq \frac{\alpha^{n+\sqrt{n}}}{\alpha^{n+1}} = \alpha^{\sqrt{n}-1}$. From this, (3.7), and Lemma 2.1, we obtain that $(n+1)\log\alpha \geq \log L_n > \alpha^{\sqrt{n}-1}$ but $(n+1)\log\alpha \leq \alpha^{\sqrt{n}-1}$ for all $n \geq 111$. So this is a contradiction. Hence, n is a prime and $n \geq 7$.

We write $L_n = {q_1}^{\gamma_1} \cdots {q_t}^{\gamma_t}$ where $q_1 < \cdots < q_t$ are prime numbers and $\gamma_i \geq 1$ for $i = 1, \ldots, t$. Let $q \in \{q_1, \ldots, q_t\}$. We claim that $q \geq 2n-1$. One way to prove this is to recall the primitive divisor theorem of Carmichael and that $p \equiv \pm 1 \pmod{N}$ if p is a primitive divisor of F_N. In our situation, because n is a prime, we see that q is a primitive divisor of L_n, so it is a primitive divisor of F_{2n}. So, $q \equiv \pm 1 \pmod{2n}$ and thus, $q \geq 2n-1$ as claimed. Alternatively, we use n is prime and apply Lemmas 2.1 and 2.2 to obtain that $q \neq 2$, $q \neq 5$, $\frac{z(q)}{2} \mid n$, and so $n = \frac{z(q)}{2} \leq \frac{q+1}{2}$. Thus, $q \geq 2n-1$ as asserted. Now by Lemmas 2.6, 2.3, and (3.4),

$$\prod_{i=1}^{t}\left(1 + \frac{1}{q_i - 1}\right) = \frac{L_n}{\varphi(L_n)} > \frac{\sigma_k(L_n)}{{L_n}^k} \geq \frac{L_{\sigma_k(n)}}{{L_n}^k} = \frac{L_{1+n^k}}{{L_n}^k} > \frac{29}{18}. \tag{3.9}$$

Taking logarithms in (3.9), and using $x > \log(1+x)$ for all $x > 0$ and $\frac{1}{2(n-1)} \ge \frac{1}{q-1}$ for every $q \in \{q_1, q_2, \dots, q_t\}$, we conclude that

$$\frac{t}{2(n-1)} \ge \sum_{i=1}^{t} \frac{1}{q_i - 1} > \sum_{i=1}^{t} \log\left(1 + \frac{1}{q_i - 1}\right) > \log\frac{29}{18}. \tag{3.10}$$

Therefore, $t > 2(n-1)\log(29/18)$. Then,

$$(n+1)\log\alpha \ge \log L_n \ge \sum_{i=1}^{t} \log q_i \ge t\log(2n-1) > 2(n-1)\log(29/18)\log(2n-1), \tag{3.11}$$

which is a contradiction. Hence, inequality (3.4) is not true, that is, $\sigma_k(L_n) < L_{\sigma_k(n)}$. □

It remains to consider the inequality between $\varphi(L_n)$ and $L_{\varphi(n)}$ as follows.

Theorem 3.4. *We have $\varphi(L_n) \ge L_{\varphi(n)}$, for all $n \ge 1$ except when $n = 3$, where $\varphi(L_3) < L_{\varphi(3)}$. In addition, $\varphi(L_n) = L_{\varphi(n)}$ if and only if $n = 1$.*

Proof. We first ran a computation to verify the result for $n \le 110$. We assume throughout that $n \ge 111$. Suppose for a contradiction that

$$L_{\varphi(n)} \ge \varphi(L_n). \tag{3.12}$$

Suppose n is not a prime. By Lemmas 2.1 and 2.4, and (3.12), we obtain

$$\alpha^{n-\varphi(n)-2} \le \frac{L_n}{L_{\varphi(n)}} \le \frac{L_n}{\varphi(L_n)} < \log L_n \le (n+1)\log\alpha. \tag{3.13}$$

If d is a divisor of n and $1 < d \le \sqrt{n}$, then the $\frac{n}{d}$ numbers $d, 2d, \dots, \frac{n}{d}\cdot d$ are less than or equal to n and are not coprime to n, which implies $n - \varphi(n) \ge \frac{n}{d} \ge \sqrt{n}$. From this and (3.13), we have $\alpha^{\sqrt{n}-2} \le \alpha^{n-\varphi(n)-2} < (n+1)\log\alpha$, which implies $n \le 110$, a contradiction. Hence, n is prime. We write $L_n = {q_1}^{\gamma_1}\cdots {q_t}^{\gamma_t}$ where $q_1 < \cdots < q_t$ are prime numbers and $\gamma_i \ge 1$ for $i = 1, \dots, t$. Similar to Theorem 3.3, we have $q_i \ge 2n-1$ for all i and

$$\prod_{i=1}^{t}\left(1 + \frac{1}{q_i - 1}\right) = \frac{L_n}{\varphi(L_n)} \ge \frac{L_n}{L_{\varphi(n)}} = \frac{L_n}{L_{n-1}} > \frac{29}{18},$$

which leads to (3.10) and (3.11). So, we have a contradiction. This completes the proof. □

Comments and Open Questions. By Pongsriiam's result [18, Lemma 2.5] on $\omega(F_n)$, it should be possible to obtain the inequality between $\omega(F_n)$ and $F_{\omega(n)}$ but the one corresponding to $\omega(L_n)$ and $L_{\omega(n)}$ seems more complicated. The question concerning $\sigma_0(L_n)$ and $L_{\sigma_0(n)}$ has not been answered. Let $\ell(n)$ be the length of longest arithmetic progressions in the least positive reduced residue system modulo n. Although we know an exact formula for $\ell(n)$ for all $n \in \mathbb{N}$ (see Pongsriiam [19]), it is not completely obvious what the inequality between $\ell(F_n)$, $F_{\ell(n)}$, $\ell(L_n)$, and $L_{\ell(n)}$ should be. Let $P(n)$ be the largest prime factor of n. Stewart [26] has recently given a new result on $P(F_n)$. Can we use Stewart's result and others to obtain the inequalities between $P(F_n)$, $F_{P(n)}$, $P(L_n)$, and $L_{P(n)}$? There are many other arithmetic functions that we may consider. We leave these problems as future research and we do not mind if the interested reader solves them.

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