REPDIGITS AS PRODUCTS OF BALANCING AND LUCAS-BALANCING NUMBERS WITH INDICES IN ARITHMETIC PROGRESSIONS

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Abstract. A repdigit is a natural number formed by the repetition of a single digit in a positional number system. This paper addresses the presence of repdigits in the products of balancing and Lucas-balancing numbers having indices in arithmetic progressions.

1. Introduction

A repdigit is a natural number N expressible in the form $N = a\left(\frac{10^m-1}{9}\right)$ for some $m \geq 1$ and $1 \leq a \leq 9$. In addition, if $a = 1$, then the repdigits are called repunits. Although the infinitude of repdigits is well-known, their presence in binary recurrence sequences is doubtful. This motivates researchers to explore all repdigits in such sequences.

Luca [3] showed that $F_{10} = 55$ and $L_5 = 11$ are the largest repdigits in the Fibonacci and Lucas sequences, respectively. Furthermore, Faye and Luca [2] proved that $P_3 = 5$ and $Q_2 = 6$ are the largest repdigits in the Pell and Pell-Lucas sequences, respectively. Subsequently, Marques and Togbé [4] proved that the product of two or more consecutive Fibonacci numbers can never be a repdigit consisting of at least two digits. In a recent paper [9], the authors explored repdigits that are balancing numbers, Lucas-balancing numbers, or products of consecutive balancing or Lucas-balancing numbers. In this paper, we explore repdigits that are expressible as products of balancing and Lucas-balancing numbers with their indices in arithmetic progressions.

2. Preliminaries

A natural number n is a balancing number if it satisfies the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$$

for some positive integer r, known as the balancer corresponding to n. For each balancing number B, $8B^2 + 1$ is a perfect square and $C = \sqrt{8B^2+1}$ is a Lucas-balancing number. The sequence of balancing numbers $(B_n)_{n\geq 0}$ and Lucas-balancing numbers $(C_n)_{n\geq 0}$ satisfy the binary recurrence $x_n = 6x_{n-1} - x_{n-2}$, $n \geq 2$ with initial terms $B_0 = 0$, $B_1 = 1$, and $C_0 = 1$, $C_1 = 3$, respectively (see [1, 6, 8, 10]). In this section, we discuss some results concerning balancing numbers, repdigits, and linear congruences. We will refer back to this section when necessary.

Theorem 2.1. ([6],Theorem 2.8). *If m and n are positive integers, then $B_m \mid B_n$ if and only if $m \mid n$.*

Theorem 2.2. *If n is a positive integer, then $3 \mid B_n$ if and only if $2 \mid B_n$.*

Proof. Modulo 3, the period of the balancing sequence is 4 and $B_n \equiv 0 \pmod 3$ if and only if $n \equiv 0, 2 \pmod 4$. Thus, $3 \mid B_n$ if and only if $2 \mid n$. Because every second term of the balancing sequence is even, the result follows. □

Theorem 2.3. *If k and n are positive integers, then $2^k \mid B_n$ if and only if $2^k \mid n$.*

Proof. Modulo 2^k, the period of the balancing sequence is 2^k (see [7]) and $B_n \equiv 0 \pmod{2^k}$ if and only if $n \equiv 0 \pmod{2^k}$. Thus, $2^k \mid B_n$ if and only if $2^k \mid n$. □

The following lemma is a basic result on the solvability of linear congruence and can be found in [5, p. 77].

Lemma 2.4. *If a, b, and n are integers, $n \geq 1$, and $\gcd(a, n) = d$, then the congruence $ax \equiv b \pmod{n}$ is solvable if and only if $d \mid b$.*

In the following lemma, we present some congruence results regarding repunits.

Lemma 2.5. *If m is a positive integer, then*
(a) $3 \mid \frac{10^m-1}{9}$ *if and only if* $3 \mid m$.
(b) $7 \mid \frac{10^m-1}{9}$ *if and only if* $6 \mid m$.

The proof of this lemma is easy and is omitted.

Lemma 2.6. *If m is a positive integer, then $\frac{10^m-1}{9}$ is periodic modulo 35 with period 6 and the least nonnegative residues are 1, 11, 6, 26, 16, 21.*

3. Main Results

In this section, we prove some theorems ascertaining the absence of repdigits in certain products of balancing numbers. The balancing numbers appearing in these products are such that their indices form arithmetic progressions. With similar types of indices, we also prove that no product of Lucas-balancing numbers is a repdigit.

The first three theorems show that products of balancing numbers with indices in arithmetic progression cannot be repdigits for a large class of common differences.

Theorem 3.1. *If m, n, k, d, and a are natural numbers, $m \geq 2$, $1 \leq a \leq 9$, $\gcd(d, 3) = 1$, and $d \not\equiv \pm 23 \pmod{60}$, then the Diophantine equation*

$$B_n B_{n+d} B_{n+2d} \cdots B_{n+kd} = a\Big(\frac{10^m - 1}{9}\Big) \tag{3.1}$$

has no solution.

Proof. First of all, we will show that (3.1) has no solution if $k \geq 2$. Since $3 \nmid d$, it follows that one of n, $n+d$, and $n+2d$ is divisible by 3 and hence, by Theorem 2.1, $B_3 = 35 \mid B_n B_{n+d} B_{n+2d}$. Thus,

$$35 \mid B_n B_{n+d} B_{n+2d} \cdots B_{n+kd} = a\Big(\frac{10^m - 1}{9}\Big).$$

If $a \in \{1, 2, 3, 4, 6, 7, 8, 9\}$, then $35 \mid \frac{10^m-1}{9}$ implies that $5 \mid \frac{10^m-1}{9}$, which is a contradiction. If $a = 5$, then $7 \mid \frac{10^m-1}{9}$ and in view of Lemma 2.5, $m \equiv 0 \pmod{6}$, which is possible only if $3 \mid m$. But, $3 \mid m$ implies that $3 \mid \frac{10^m-1}{9}$ and hence, $3 \mid B_n B_{n+d} B_{n+2d} \cdots B_{n+kd}$. So $3 \mid B_{n+id}$ for some $0 \leq i \leq k$ and in view of Theorem 2.2, $2 \mid B_{n+id}$ and consequently, $2 \mid 5 \cdot \frac{10^m-1}{9}$, which is a contradiction. Hence, $35 \nmid a \cdot \frac{10^m-1}{9}$ for any $a \in \{1, 2, \ldots, 9\}$ and so (3.1) has no solution for $k \geq 2$.

Finally, if $k = 1$, (3.1) reduces to

$$B_n B_{n+d} = a\Big(\frac{10^m - 1}{9}\Big).$$

For the proof, we need the periodic least residues of the sequence of balancing numbers modulo 4, 5, 7, 8, 9, 11, 20, and 100. We list them in Table 1.

Row no.	m	$B_n \bmod m$	Period
1	4	$0, 1, 2, 3$	4
2	5	$0, 1, 1, 0, 4, 4$	6
3	7	$0, 1, 6$	3
4	8	$0, 1, 6, 3, 4, 5, 2, 7$	8
5	9	$0, 1, 6, 8, 6, 1, 0, 8, 3, 1, 3, 8$	12
6	11	$0, 1, 6, 2, 6, 1, 0, 10, 5, 9, 5, 10$	12
7	20	$0, 1, 6, 15, 4, 9, 10, 11, 16, 5, 14, 19$	12
8	100	$0, 1, 6, 35, 4, 89, 30, 91, 16, 5, 14, 79, 60, 81, 26, 75, 24, 69, 90, 71,$ $36, 45, 34, 59, 20, 61, 46, 15, 44, 49, 50, 51, 56, 85, 54, 39, 80, 41, 66, 55,$ $64, 29, 10, 31, 76, 25, 74, 19, 40, 21, 86, 95, 84, 9, 70, 11, 96, 65, 94, 99$	60

TABLE 1

To complete the proof, we discuss eight cases corresponding to the values of d.

Case 1: $d \equiv \pm 1 \pmod{12}$. Modulo 20, the least residues of $B_nB_{n+d} \in \{0, 6, 10, 16\}$, from which it follows that $a \notin \{1, 2, 3, 4, 5, 7, 8, 9\}$. So, the only possible case is $a = 6$. We will show that $a = 6$ is also not possible. For this, it is sufficient to show that $B_nB_{n+d} \not\equiv 66 \pmod{100}$. If $d \equiv \pm 1 \pmod{12}$, then $d \equiv \pm 1, \pm 11, \pm 13, \pm 23, \pm 25 \pmod{60}$. For $d \equiv k \pmod{60}$, we list the least residues of $B_nB_{n+d} \pmod{100}$ in Table 2.

Row no.	k	$B_nB_{n+d} \bmod 100$ belongs to
1	± 1	$\{0, 6, 10, 20, 30, 40, 50, 56, 60, 70, 80, 90\}$
2	± 11	$\{0, 10, 20, 30, 36, 40, 50, 60, 70, 80, 86, 90\}$
3	± 13	$\{0, 10, 20, 26, 30, 40, 50, 60, 70, 76, 80, 90\}$
4	± 23	$\{0, 10, 16, 20, 30, 40, 50, 60, 66, 70, 80, 90\}$
5	± 25	$\{0, 10, 20, 30, 40, 46, 50, 60, 70, 80, 90, 96\}$

TABLE 2

It follows from the above discussion and Table 1 and Table 2 that (3.1) has no solution if $m \geq 2$, $1 \leq a \leq 9$, and $d \equiv k \pmod{60}$, where $k \in \{1, 11, 13, 25, 35, 47, 49, 59\}$.

Case 2: $d \equiv 2 \pmod{12}$. The least residues of B_nB_{n+d} in a period modulo 20 are 0, 15, 4, 15, 0, 19, from which it is clear that $a \notin \{1, 2, 3, 6, 7, 8\}$ or $a \in \{4, 5, 9\}$.

If $a = 4$, then $B_nB_{n+d} \equiv 0 \pmod 4$. But, the least residues of B_nB_{n+d} in a period modulo 4 are 0, 3 and hence, n is even. Since $d \equiv 2 \pmod{12}$, by virtue of Theorem 2.3, we have $B_nB_{n+d} \equiv 0 \pmod 8$, i.e., $2 \mid \frac{10^m-1}{9}$, which is a contradiction.

If $a = 5$, then $B_nB_{n+d} \equiv 0 \pmod 5$ and since the least residues of B_nB_{n+d} in a period modulo 5 are 0, 0, 4, it follows that $n \equiv 0, 1 \pmod 3$. If $n \equiv 0 \pmod 3$, then $3 \mid n$ and if $n \equiv 1 \pmod 3$, then $3 \mid (n + d)$, implying $B_3 \mid B_nB_{n+d}$, which is a contradiction. It can also be observed the least residues of B_nB_{n+d} in a period modulo 7 are 0, 0, 6 and thus, if $n \equiv 1 \pmod 3$, then $B_nB_{n+d} \equiv 0 \pmod 7$. Hence, $7 \mid \frac{10^m-1}{9}$, which is a contradiction.

If $a = 9$, then $B_nB_{n+d} \equiv 0 \pmod 9$. But, the least residues of B_nB_{n+d} in a period modulo 9 are 0, 8, which implies that $2 \mid n$. Thus, $2 \mid 9 \cdot \frac{10^m-1}{9}$, which is impossible.

Case 3: $d \equiv 4 \pmod{12}$. The least residues of B_nB_{n+d} in a period modulo 20 are 0, 9, 0, 5, 4, 5 from which it is clear that $a \notin \{1, 2, 3, 5, 6, 7, 8, 9\}$. So, the only possible case is $a = 4$. But, $B_nB_{n+d} \equiv 4 \pmod{20}$ implies $n \equiv 4 \pmod 6$. Since the least residues of B_nB_{n+d} in a period modulo 11 are 0, 1, 0, 9, 8, 9, it follows if $n \equiv 4 \pmod 6$, then $B_nB_{n+d} \equiv 8 \pmod{11}$. But, this implies $\frac{10^m-1}{9} \equiv 2 \pmod{11}$, which is a contradiction.

Case 4: $d \equiv 5 \pmod{12}$. The least residues of B_nB_{n+d} in a period modulo 20 are 0, 10, 6, 0, 0, 6, 10, 0, 16, 10, 10, 16 from which it is clear that $a \notin \{1, 2, 3, 4, 5, 7, 8, 9\}$. So, the only possible case is $a = 6$. But, $B_nB_{n+d} \equiv 6 \pmod{20}$ implies $n \equiv 2, 5 \pmod{12}$. Since the least residues of B_nB_{n+d} in a period modulo 11 are 0, 0, 5, 10, 10, 5, it follows if $n \equiv 2, 5 \pmod{12}$, then $B_nB_{n+d} \equiv 5 \pmod{11}$, which implies $\frac{10^m-1}{9} \equiv 10 \pmod{11}$, which is a contradiction.

Case 5: $d \equiv 7 \pmod{12}$. The least residues of B_nB_{n+d} in a period modulo 20 are 0, 16, 10, 10, 16, 0, 10, 6, 0, 0, 6, 10, 0 and therefore, $a \notin \{1, 2, 3, 4, 5, 7, 8, 9\}$. So, the only possible case is $a = 6$. But, $B_nB_{n+d} \equiv 6 \pmod{20}$ implies that $n \equiv 7, 10 \pmod{12}$. Since the least residues of B_nB_{n+d} in a period modulo 11 are 0, 5, 10, 10, 5, 0, it follows that, if $n \equiv 7, 10 \pmod{12}$, then $B_nB_{n+d} \equiv 5 \pmod{11}$, which implies that $\frac{10^m-1}{9} \equiv 10 \pmod{11}$, which is not true.

Case 6: $d \equiv 8 \pmod{12}$. The least residues of B_nB_{n+d} in a period modulo 20 are 0, 5, 4, 5, 0, 9 and hence, $a \notin \{1, 2, 3, 5, 6, 7, 8, 9\}$. So, the only possible case is $a = 4$. But, $B_nB_{n+d} \equiv 4 \pmod{20}$ implies $n \equiv 2 \pmod 6$. Since the least residues of B_nB_{n+d} in a period modulo 11 are 0, 9, 8, 9, 0, 1, it follows if $n \equiv 2 \pmod 6$, then $B_nB_{n+d} \equiv 8 \pmod{11}$, which implies $\frac{10^m-1}{9} \equiv 2 \pmod{11}$, which is not possible.

Case 7: $d \equiv 10 \pmod{12}$. The least residues of B_nB_{n+d} in a period modulo 20 are 0, 19, 0, 15, 4, 15, so $a \notin \{1, 2, 3, 6, 7, 8\}$ or $a \in \{4, 5, 9\}$.

If $a = 4$, then $B_nB_{n+d} \equiv 4 \pmod{20}$, which implies $n \equiv 4 \pmod 6$. Since the least residues of B_nB_{n+d} in a period modulo 11 are 0, 10, 0, 2, 3, 2, 0, 10, 0, 2, 3, 2, 0, it follows if $n \equiv 4 \pmod 6$, then $B_nB_{n+d} \equiv 3 \pmod{11}$, which implies $\frac{10^m-1}{9} \equiv 9 \pmod{11}$. This is not true.

If $a = 5$, then $B_nB_{n+d} \equiv 0 \pmod 5$, and since the least residues of B_nB_{n+d} in a period modulo 5 are 0, 4, 0, it follows that $n \equiv 0, 2 \pmod 3$. If $n \equiv 0 \pmod 3$, then $3 \mid n$ and if $n \equiv 2 \pmod 3$, then $3 \mid (n + d)$, implying $B_3 \mid B_nB_{n+d}$, which is a contradiction. It can also be observed that the least residues of B_nB_{n+d} in a period modulo 7 are 0, 6, 0 from which it follows if $n \equiv 2 \pmod 3$, then $B_nB_{n+d} \equiv 0 \pmod 7$. Hence, $7 \mid \frac{10^m-1}{9}$, which is a contradiction.

If $a = 9$, then $B_nB_{n+d} \equiv 0 \pmod 9$ and since the least residues of B_nB_{n+d} in a period modulo 9 are 0, 8, it follows that $2 \mid n$. Thus, $2 \mid 9 \cdot \frac{10^m-1}{9}$, which is impossible.

It follows from the above discussion that (3.1) has no solution if $m \geq 2$, $1 \leq a \leq 9$, and $d \not\equiv \pm 23 \pmod{60}$. This completes the proof. □

Remark. It is clear if $m = 1$, then the only solution of (3.1) is $(n, k, d, a) = (1, 1, 1, 6)$ i.e., $B_1B_2 = 6$.

Theorem 3.2. *If m, n, k, d, and a are natural numbers, $1 \leq a \leq 9$, $a \neq 6$, and $d \equiv \pm 23 (\text{mod } 60)$, then the Diophantine equation*

$$B_n B_{n+d} B_{n+2d} \cdots B_{n+kd} = a\left(\frac{10^m - 1}{9}\right) \tag{3.2}$$

has no solution.

Proof. If $d \equiv \pm 23 (\text{mod } 60)$, then $d \equiv \pm 1 \pmod{12}$. The proof follows from the proof of Theorem 3.1. □

In Theorems, 3.1 and 3.2, we discussed the existence of repdigits that are products of balancing numbers with indices in arithmetic progression, and the common differences are not divisible by 3. In what follows, we discuss the existence of such repdigits where the common differences are multiples of 3.

Theorem 3.3. *If m, n, k, d, and a are natural numbers and $1 \leq a \leq 9$, then the Diophantine equation*

$$B_n B_{n+3d} B_{n+6d} \cdots B_{n+3kd} = a\left(\frac{10^m - 1}{9}\right) \tag{3.3}$$

has no solution if one of the following condition holds:

(1) $3 \mid n$
(2) *d is even*
  (a) *k is odd, $a \notin \{1, 6\}$*
  (b) *k is even*
    (i) $a \notin \{1, 6\}$, $n \equiv 1, 2 \pmod 6$
    (ii) $a \notin \{4, 9\}$, $n \equiv 4 \pmod 6$
    (iii) $a \neq 4$, $n \equiv 5 \pmod 6$
(3) *d is odd*
  (a) $k \equiv 0 \pmod 4$
    (i) $a \notin \{1, 6\}$, $n \equiv 1, 2 \pmod 6$
    (ii) $a \notin \{4, 9\}$, $n \equiv 4 \pmod 6$
    (iii) $a \neq 4$, $n \equiv 5 \pmod 6$
  (b) $k \equiv 1 \pmod 4$, $a \notin \{4, 9\}$
  (c) $k \equiv 2 \pmod 4$
    (i) $a \notin \{4, 9\}$, $n \equiv 1 \pmod 6$
    (ii) $a \neq 4$, $n \equiv 2 \pmod 6$
    (iii) $a \notin \{1, 6\}$, $n \equiv 4, 5 \pmod 6$
  (d) $k \equiv 3 \pmod 4$, $a \notin \{1, 6\}$

Proof. In view of Theorem 2.1, if $3 \mid n$, then $B_3 \mid B_n B_{n+3d} B_{n+6d} \cdots B_{n+3kd}$ and consequently, $B_3 \mid a\left(\frac{10^m-1}{9}\right)$. But, while proving Theorem 3.1, we saw no repdigit is a multiple of $B_3 = 35$. This completes the proof of (1).

Modulo 35, the least residues of B_n are 0, 1, 6, 0, 29, 34 and in view of (3.3),

$$B_n B_{n+3d} B_{n+6d} \cdots B_{n+3kd} = a\left(\frac{10^m - 1}{9}\right) \equiv 1, 6, 29, 34 \pmod{35}. \tag{3.4}$$

In view of Lemma 2.4, $a\left(\frac{10^m-1}{9}\right) \equiv 1, 6, 29, 34 \pmod{35}$ has a solution if and only if $a \in \{1, 2, 3, 4, 6, 8, 9\}$. Using the least residues of the repunits modulo 35 (see Lemma 2.6) and the restrictions over d, k, n, and a in (3.4), the other proofs are similar to Theorem 3.1. □

The following is an immediate consequence of the above results.

Corollary 3.4. *If* m, n, k, *and* d *are natural numbers and* $a \in \{2, 3, 5, 7, 8\}$, *then the Diophantine equation*

$$B_n B_{n+d} B_{n+2d} \cdots B_{n+kd} = a\Big(\frac{10^m - 1}{9}\Big)$$

has no solution.

Next, we discuss the existence of repdigits in the products of Lucas-balancing numbers with indices in arithmetic progression.

Theorem 3.5. *If* m, n, k, d, *and* a *are natural numbers,* $1 \le a \le 9$, *then the Diophantine equation*

$$C_n C_{n+d} C_{n+2d} \cdots C_{n+kd} = a\Big(\frac{10^m - 1}{9}\Big) \tag{3.5}$$

has no solution.

Proof. Because Lucas-balancing numbers are odd, (3.5) implies $a \in \{1, 3, 5, 7, 9\}$. It is easy to see (3.5) has no solution if $m = 1, 2$. To complete the proof, we need the least residues of Lucas-balancing numbers and their products modulo 5, 7, and 8. We list them in Table 3.

m	$C_n \mod m$	$C_n C_{n+d} \cdots C_{n+kd} \mod m$ belongs to
5	$1, 3, 2, 4, 2, 3$	$\{1, 2, 3, 4\}$
7	$1, 3, 3$	$\{1, 2, 3, 4, 5, 6\}$
8	$1, 3$	$\{1, 3\}$

Table 3

For $m \ge 3$, (3.5) implies that $C_n C_{n+d} C_{n+2d} \cdots C_{n+kd} = a\Big(\frac{10^m-1}{9}\Big) \equiv 7a \pmod 8$. A quick look at the last row of Table 3 implies $7a \equiv 1, 3 \pmod 8$ so $a \in \{5, 7\}$. Furthermore, $C_n C_{n+d} C_{n+2d} \cdots C_{n+kd} \equiv 0 \pmod a$. But, 0 is not a least residue of $C_n C_{n+d} C_{n+2d} \cdots C_{n+kd}$ modulo 5 or 7 and hence, (3.5) has no solution for $m \ge 3$. This completes the proof. ☐

4. Conclusion

In this work, we showed no product of Lucas-balancing numbers with indices in arithmetic progression is a repdigit and, as far as balancing numbers are concerned, similar products of balancing numbers cannot be repdigits in most cases. However, we failed to prove the existence or nonexistence of repdigits of the form if $d \not\equiv \pm 23 \pmod{60}$. We also failed to handle some cases when $d \equiv 0 \pmod 3$. We leave these as open problems for the interested researchers.

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